Solutions to Review Problems for Final Exam

1. In this problem, u and v denote vectors in $\mathbb{R}^n$.

   (a) Use the triangle inequality to derive the inequality

   $$|\ \|v\| - \|u\|\ | \leqslant \|v - u\| \quad \text{for all} \quad u, v \in \mathbb{R}^n. \tag{1}$$

   Solution: Write
   $$\|u\| = \|(u - v) + v\|$$
   and applying the triangle inequality to obtain

   $$\|u\| \leqslant \|u - v\| + \|v\|,$$

   from which we get that

   $$\|u\| - \|v\| \leqslant \|v - u\|. \tag{2}$$

   Interchanging the roles for u and v in (2) we obtain

   $$\|v\| - \|u\| \leqslant \|u - v\|.$$

   from which we get
   $$\|v\| - \|u\| \leqslant \|v - u\|. \tag{3}$$
   Combining (2) and (3) yields

   $$-\|v - u\| \leqslant \|v\| - \|u\| \leqslant \|v - u\|,$$

   which is (1). □

   (b) Use the inequality derived in the previous part to show that the function $f\colon \mathbb{R}^n \to \mathbb{R}$ given by $f(v) = \|v\|$, for all $v \in \mathbb{R}^n$, is continuous in $\mathbb{R}^n$.

   Solution: Fix $u \in \mathbb{R}^n$ and apply the inequality in (1) to any $v \in \mathbb{R}^n$ to obtain that
   $$|\|v\| - \|u\|| \leqslant \|v - u\|,$$
   or
   $$|f(v) - f(u)| \leqslant \|v - u\|. \tag{4}$$
   Next, apply the Squeeze Lemma to obtain from (4) that

   $$\lim_{\|v-u\|\to 0} |f(v) - f(u)| = 0,$$

   which shows that f is continuous at u for any $u \in \mathbb{R}^n$. □

(c) Prove that the function $g\colon \mathbb{R}^n \to \mathbb{R}$ given by $g(v) = \sin(\|v\|)$, for all$v \in \mathbb{R}^n$, is continuous.

***Solution**:* Observe that $g = \sin \circ f$, where $f\colon \mathbb{R}^n \to \mathbb{R}$ is as defined in part (b). Thus, g is the composition of two continuous functions, and is, therefore, continuous. $\square$

2. Define the scalar field $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(v) = \|v\|^2$ for all $v \in \mathbb{R}^n$.

(a) Show that f is differentiable in $\mathbb{R}^n$ and compute the linear map

$$Df(u)\colon \mathbb{R}^n \to \mathbb{R} \quad \text{for all} \quad u \in \mathbb{R}^n.$$

What is the gradient of f at u for all $u \in \mathbb{R}^n$?

***Solution**:* Let $u \in \mathbb{R}^n$ and compute

$$\begin{aligned} f(u+w) &= \|u+w\|^2 \\ &= (u+w)\cdot(u+w) \\ &= u\cdot u + u\cdot w + w\cdot u + w\cdot w \\ &= \|u\|^2 + 2u\cdot w + \|w\|^2, \end{aligned}$$

for $w \in \mathbb{R}^n$, where we have used the symmetry of the dot product and the fact that $\|v\|^2 = v\cdot v$ for all $v \in \mathbb{R}^n$. We therefore have that

$$f(u+w) = f(u) + 2u\cdot w + \|w\|^2, \qquad \text{for all } u \in \mathbb{R}^n \text{ and } w \in \mathbb{R}^n. \tag{5}$$

Writing

$$Df(u)w = 2u\cdot w, \qquad \text{for all } u \in \mathbb{R}^n \text{ and } w \in \mathbb{R}^n, \tag{6}$$

and

$$E_u(w) = \|w\|^2, \qquad \text{for all } u \in \mathbb{R}^n \text{ and } w \in \mathbb{R}^n, \tag{7}$$

we see that (5) can be rewritten as

$$f(u+w) = f(u) + Df(u)w + E_u(w), \qquad \text{for all } u \in \mathbb{R}^n \text{ and } w \in \mathbb{R}^n, \tag{8}$$

where, according to (6), $Df(u)\colon \mathbb{R}^n \to \mathbb{R}^n$ defines a linear transformation, and, by virtue of (7),

$$\frac{|E_u(w)|}{\|w\|} = \|w\|, \qquad \text{for } w \neq 0,$$

from which we get that

$$\lim_{\|w\|\to 0}\frac{|E_u(w)|}{\|w\|} = 0.$$

Consequently, in view of (8), we conclude that f is differentiable at every $u\in\mathbb{R}^n$, derivative at u given by (6).

Since $Df(u)w = \nabla f(u)\cdot w$, for all u and w in $\mathbb{R}^n$, by comparing with (6), we see that

$$\nabla f(u) = 2u, \quad \text{for all } u\in\mathbb{R}^n.$$

□

Alternate Solution: Alternatively, for $u = (x_1, x_2, \ldots, x_n)\in\mathbb{R}^n$, we have that

$$f(u) = x_1^2 + x_2^2 + \cdots + x_n^2;$$

so that

$$\frac{\partial f}{\partial x_j}(u) = 2x_j, \quad \text{for } j = 1, 2, \ldots, n.$$

Thus, all the partial derivatives of f are continuous on $\mathbb{R}^n$; that is, f is a C^1 function. Consequently, f is differentiable on $\mathbb{R}^n$. Furthermore,

$$Df(u)w = \begin{pmatrix} 2x_1 & 2x_2 & \cdots & 2x_n\end{pmatrix} w, \quad \text{for all } w\in\mathbb{R}^n,$$

which can be written as

$$Df(u)w = 2u\cdot w, \quad \text{for all } w\in\mathbb{R}^n. \tag{9}$$

It then follows that $\nabla f(u) = 2u$ for all $u\in\mathbb{R}^n$. □

(b) Let $\widehat{v}$ denote a unit vector in $\mathbb{R}^n$. For a fixed vector u in $\mathbb{R}^n$, define $g\colon\mathbb{R}\to\mathbb{R}$ by $g(t) = \|u - t\widehat{v}\|^2$, for all $t\in\mathbb{R}$. Show that g is differentiable and compute $g'(t)$ for all $t\in\mathbb{R}$.

Solution: Observe that $g = f\circ\sigma$, where $\sigma\colon\mathbb{R}\to\mathbb{R}^n$ is given by

$$\sigma(t) = u - t\widehat{v}, \quad \text{for all } t\in\mathbb{R}. \tag{10}$$

Thus, σ is a differentiable path with

$$\sigma'(t) = -\widehat{v}, \quad \text{for all } t\in\mathbb{R}. \tag{11}$$

Thus, by the result from part (a), g is the composition of two differentiable functions. Consequently, by the Chain Rule, g is differentiable with

$$g'(t) = Df(\sigma(t))\sigma'(t), \quad \text{for all } t\in\mathbb{R}. \tag{12}$$

Thus, using (9) and (11), we obtain from (12) that

$$g'(t) = 2\sigma(t) \cdot (-\widehat{v}), \quad \text{for all } t \in \mathbb{R},$$

or

$$g'(t) = -2\sigma(t) \cdot \widehat{v}, \quad \text{for all } t \in \mathbb{R}. \tag{13}$$

Thus, using (10), we obtain from (13) that

$$g'(t) = -2(u - t\widehat{v}) \cdot \widehat{v}, \quad \text{for all } t \in \mathbb{R},$$

which leads to

$$g'(t) = 2t - 2u \cdot \widehat{v}, \quad \text{for all } t \in \mathbb{R}, \tag{14}$$

since $\widehat{v}$ is a unit vector in $\mathbb{R}^n$. □

(c) Let $\widehat{v}$ be as in the previous part. For any $u \in \mathbb{R}^n$, give the point on the line spanned by $\widehat{v}$ which is the closest to u. Justify your answer.

***Solution**:* It follows from (14) that $g''(t) = 2 > 0$ for all $t \in \mathbb{R}$; so that g has a global minimum when $g'(t) = 0$. We then obtain from (14) that $g(t)$ is the smallest possible when

$$t = u \cdot \widehat{v}.$$

Consequently, the point on the line spanned by $\widehat{v}$ that is the closest to u is $(u \cdot \widehat{v})\widehat{v}$, or the orthogonal projection of u onto the direction of $\widehat{v}$. □

3. Let I denote an open interval which contains the real number a. Assume that $\sigma\colon I \to \mathbb{R}^n$ is a C^1 parametrization of a curve C in $\mathbb{R}^n$. Define $s\colon I \to \mathbb{R}$ as follows:

$$s(t) = \text{arlength along the curve } C \text{ from } \sigma(a) \text{ to } \sigma(t), \tag{15}$$

for all $t \in I$.

(a) Give a formula, in terms of an integral, for computing $s(t)$ for all $t \in I$.

***Answer**:*

$$s(t) = \int_a^t \|\sigma'(\tau)\| \, d\tau, \quad \text{for all } t \in I. \tag{16}$$

□

(b) Prove that s is differentiable on I and compute $s'(t)$ for all $t \in I$. Deduce that s is strictly increasing with increasing t.

***Solution**:* It follows from the assumption that σ is C^1, the Fundamental Theorem of Calculus, and (16), that s is differentiable and

$$s'(t) = \|\sigma'(t)\|, \qquad \text{for all } t \in I. \tag{17}$$

Since we are also assuming that σ is a parametrization of a C^1 curve, C, it follows that $\sigma'(t) \neq \mathbf{0}$ for all $t \in I$. Consequently, we obtain from (17) that

$$s'(t) > 0, \qquad \text{for all } t \in I,$$

which shows that $s(t)$ is strictly increasing with increasing t. □

(c) Let ℓ denote the arclength of C, and suppose that $\gamma\colon [0, \ell] \to \mathbb{R}^n$ is a a parametrization of C with the arclength parameter s defined in (15); so that,

$$C = \{\gamma(s) \mid 0 \leqslant s \leqslant \ell\}.$$

Use the fact that $\sigma(t) = \gamma(s(t))$, for all $t \in [a, b]$, to show $\gamma'(s)$ is a unit vector that is tangent to the curve C at the point $\gamma(s)$.

***Solution**:* Note that $\sigma = \gamma \circ s$ is a composition of two differentiable functions, by the result of part (b). Consequently, by the chain rule,

$$\sigma'(t) = \frac{ds}{dt}\gamma'(s), \qquad \text{for } t \in (a, b).$$

Thus, using (16),

$$\sigma'(t) = \|\sigma'(t)\|\gamma'(s), \qquad \text{for } t \in (a, b).$$

So, using the fact that $\|\sigma'(t)\| > 0$ for all $t \in (a, b)$,

$$\gamma'(s) = \frac{1}{\|\sigma'(t)\|}\, \sigma'(t), \qquad \text{for } t \in (a, b),$$

which shows that $\gamma'(s)$ is a unit vector that is tangent to the curve C at the point $\gamma(s)$. □

4. Let I denote an open interval of real numbers and $f\colon I \to \mathbb{R}$ be a differentiable function. Let $a, b \in I$ be such that $a < b$, and define C to the section of the graph of $y = f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$; that is,

$$C = \{(x, y) \in \mathbb{R}^2 \mid y = f(x) \text{ and } a \leqslant x \leqslant b\}$$

(a) By providing an appropriate parametrization of C, compute the arclenth of C, $\ell(C)$.

***Solution**:* Parametrize C by $\sigma\colon [a, b] \to \mathbb{R}^2$ given by

$$\sigma(t) = (t, f(t)), \qquad \text{for } a \leqslant t \leqslant b.$$

Then,

$$\sigma'(t) = (1, f'(t)), \qquad \text{for } a \leqslant t \leqslant b;$$

so that

$$\|\sigma'(t)\| = \sqrt{1 + [f'(t)]^2}, \qquad \text{for } a \leqslant t \leqslant b.$$

Therefore,

$$\ell(C) = \int_a^b \sqrt{1 + [f'(t)]^2}\; dt. \tag{18}$$

□

(b) Let $f(x) = 5 - 2x^{3/2}$, for $x \geqslant 0$. Compute the exact arcength of $y = f(x)$ over the interval $[0, 11]$.

***Solution**:* We use the formula in (18) with

$$f'(t) = -3t^{1/2}, \qquad \text{for } t > 0.$$

Thus,

$$\begin{aligned} \ell(C) &= \int_0^{11} \sqrt{1 + 9t}\; dt \\ &= \left[\frac{2}{27}(1 + 9t)^{3/2}\right]_0^{11} \\ &= \frac{2}{27}(1000 - 1) \\ &= 74. \end{aligned}$$

□

5. Let $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the map from the uv–plane to the xy–plane given by

$$\Phi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u \\ v^2 \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2, \tag{19}$$

and let T be the oriented triangle $[(0,0),(1,0),(1,1)]$ in the uv–plane.

(a) Show that Φ is differentiable and give a formula for its derivative, $D\Phi(u,v)$, at every point $\begin{pmatrix} u \\ v \end{pmatrix}$ in $\mathbb{R}^2$.

***Solution**:* Write

$$\Phi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2,$$

where $f(u,v) = 2u$ and $g(u,v) = v^2$ for all $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$. Observe that the partial derivatives of f and g exist and are given by

$$\frac{\partial f}{\partial u}(u,v) = 2, \qquad \frac{\partial f}{\partial v}(u,v) = 0$$

$$\frac{\partial g}{\partial u}(u,v) = 0, \qquad \frac{\partial g}{\partial v}(u,v) = 2v.$$

Note that the partial derivatives of f and g are continuous. Therefore, Φ is a C^1 map. Hence, Φ is differentiable on $\mathbb{R}^2$ and its derivative map at (u,v), for any $(u,v) \in \mathbb{R}^2$, is given by multiplication by the Jacobian matrix

$$D\Phi(u,v) = \begin{pmatrix} 2 & 0 \\ 0 & 2v \end{pmatrix};$$

that is,

$$D\Phi(u,v)\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2v \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} 2h \\ 2vk \end{pmatrix}$$

for all $\begin{pmatrix} h \\ k \end{pmatrix} \in \mathbb{R}^2$. $\square$

(b) Give the image, R, of the triangle T under the map Φ, and sketch it in the xy–plane.

***Solution**:* The image of T under Φ is the set

$$\begin{aligned} \Phi(T) &= \{(x,y) \in \mathbb{R}^2 \mid x = 2u, y = v^2, \text{ for some } (u,v) \in \mathbb{R}\} \\ &= \{(x,y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant x^2/4\}. \end{aligned}$$

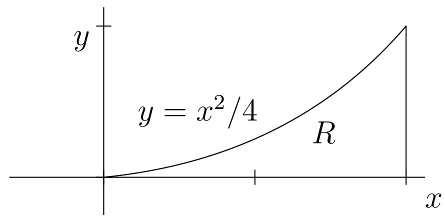

Figure 1: Sketch of Region $\Phi(T)$

A sketch of $R = \Phi(T)$ is shown in Figure 1.

To see how the sketch in Figure 1 is obtained, refer to the sketch of the triangle T in the uv–plane shown in Figure 2.

We first see where the boundary of T gets mapped by the transformation Φ. Note the boundary of T consists of three straight line segments $[(0,0),(1,0)]$, $[(1,0),(1,1)]$ and $[(1,1),(0,0)]$ oriented according a counterclockwise orientation along the boundary of T. The straight line segment $[(0,0),(1,0)]$ can be parametrized by

$$u = t, \ v = 0, \qquad \text{for } 0 \leqslant t \leqslant 1 \tag{20}$$

Applying map Φ in (19) to the parametric equations in (20) we obtain

$$\Phi \begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} 2t \\ 0 \end{pmatrix} \qquad \text{for } 0 \leqslant t \leqslant 1,$$

which yields the parametric equations

$$x = 2t, \ y = 0, \qquad \text{for } 0 \leqslant t \leqslant 1,$$

or the straight line segment $[(0,0),(2,0)]$ in the xy–plane shown in Figure 1.

Similarly, the straight line segment $[(1,0),(1,1)]$ in the uv–plane can be parametrized by the equations

$$u = 1, \ v = t, \qquad \text{for } 0 \leqslant t \leqslant 1.$$

Applying the map Φ yields

$$\Phi \begin{pmatrix} 1 \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ t^2 \end{pmatrix} \qquad \text{for } 0 \leqslant t \leqslant 1;$$

so that, the straight line segment $[(1,0),(1,1)]$ in the uv–plane gets mapped to the a set in the xy–plane with parametric equations

$$x = 2, \ y = t^2, \qquad \text{for } 0 \leqslant t \leqslant 1,$$

which is a parametrization of the straight line segment $[(2,0),(2,1)]$ shown in Figure 1.

Finally, the segment $[(1,1),(0,0)]$ in the uv–plane can be parametrized by

$$u = 1-t, \ v = 1-t, \qquad \text{for } 0 \leqslant t \leqslant 1.$$

Applying the mapping Φ then yields

$$\Phi\begin{pmatrix} 1-t \\ 1-t \end{pmatrix} = \begin{pmatrix} 2(1-t) \\ (1-t)^2 \end{pmatrix} \qquad \text{for } 0 \leqslant t \leqslant 1,$$

from which we get the parametric equations

$$x = 2(1-t)^2, y = (1-t)^2, \qquad \text{for } 0 \leqslant t \leqslant 1. \tag{21}$$

Squaring on both sides of the first equation in (21) yields

$$x^2 = 4(1-t)^2, \qquad \text{for } 0 \leqslant t \leqslant 1,$$

and, comparing this with the second equation in (21), we obtain

$$x^2 = 4y,$$

from which we get that

$$y = \frac{1}{4}x^2, \qquad \text{for } 0 \leqslant x \leqslant 2. \tag{22}$$

we have therefore shown that the function Φ maps the straight line segment $[(1,1),(0,0)]$ to the portion of the graph of the equation in (22) from $(0,0)$ to $(2,1)$ in the xy–plane as shown in Figure 1. □

(c) Evaluate the integral $\displaystyle\iint_R dxdy$, where R is the region in the xy–plane obtained in part (b).

Solution: Compute by means of iterated integrals

$$\begin{aligned}\iint_R dxdy &= \int_0^2 \int_0^{x^2/4} dy\ dx \\ &= \int_0^2 \frac{x^2}{4}\ dx \\ &= \left[\frac{x^3}{12}\right]_0^2 \\ &= \frac{2}{3}.\end{aligned}$$

□

(d) Evaluate the integral $\displaystyle\iint_T |\det[D\Phi(u,v)]|\ dudv$, where $\det[D\Phi(u,v)]$ denotes the determinant of the Jacobian matrix of Φ obtained in part (a). Compare the result obtained here with that obtained in part (c).

Solution: Compute $\det[D\Phi(u,v)]$ to get

$$\det[D\Phi(u,v)] = 4v.$$

so that

$$\iint_T |\det[D\Phi(u,v)]| dudv = \iint_T 4|v|\ dudv,$$

where the region T, in the uv–plane is sketched in Figure 2. Observe that,

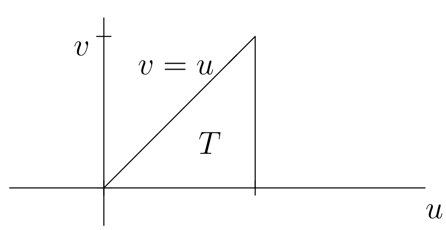

Figure 2: Sketch of Region T

in that region, $v \geqslant 0$, so that

$$\iint_T |\det[D\Phi(u,v)]| dudv = \iint_T 4v\ dudv,$$

Compute by means of iterated integrals

$$\begin{aligned}\iint_T |\det[D\Phi(u,v)]| dudv &= \int_0^1 \int_0^u 4v\ dvdu \\ &= \int_0^1 2u^2\ du \\ &= \frac{2}{3},\end{aligned}$$

which is the same result as that obtained in part (c). □

6. Consider the iterated integral $\displaystyle\int_0^1 \int_{x^2}^1 x\sqrt{1-y^2}\ dydx$.

   (a) Identify the region of integration, R, for this integral and sketch it.
   Solution: The region $R = \{(x,y) \in \mathbb{R}^2 \mid x^2 \leqslant y \leqslant 1,\ 0 \leqslant x \leqslant 1\}$ is sketched in Figure 3. □

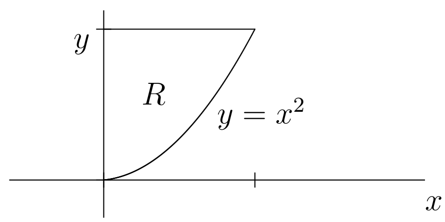

Figure 3: Sketch of Region R

   (b) Change the order of integration in the iterated integral and evaluate the double integral $\displaystyle\int_R x\sqrt{1-y^2}\ dxdy$.
   Solution: Compute

$$\begin{aligned}\iint_R x\sqrt{1-y^2}\ dxdy &= \int_0^1 \int_0^{\sqrt{y}} x\sqrt{1-y^2}\ dxdy \\ &= \int_0^1 \left[\frac{x^2}{2}\sqrt{1-y^2}\right]_0^{\sqrt{y}} dy \\ &= \int_0^1 \frac{y}{2}\sqrt{1-y^2}\ dy.\end{aligned}$$

Next, make the change of variables $u = 1 - y^2$ to obtain that

$$\begin{aligned}\iint_R x\sqrt{1-y^2}\; dxdy &= -\frac{1}{4}\int_1^0 \sqrt{u}\; du \\ &= \frac{1}{4}\int_0^1 \sqrt{u}\; du \\ &= \frac{1}{6}.\end{aligned}$$

□

7. Let $f\colon \mathbb{R} \to \mathbb{R}$ denote a twice–differentiable real valued function and define

$$u(x,t) = f(x - ct) \quad \text{for all} \quad (x,t) \in \mathbb{R}^2,$$

where c is a real constant.

Verify that $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$.

***Solution**:* Apply the chain rule to obtain

$$\frac{\partial u}{\partial x} = f'(x-ct)\cdot\frac{\partial}{\partial x}(x-ct) = f'(x-ct).$$

Similarly,

$$\frac{\partial^2 u}{\partial x^2} = f''(x-ct), \tag{23}$$

$$\frac{\partial u}{\partial t} = f'(x-ct)\cdot\frac{\partial}{\partial t}(x-ct) = -cf'(x-ct),$$

and

$$\frac{\partial^2 u}{\partial t^2} = c^2 f''(x-ct). \tag{24}$$

Combining (23) and (24) we see that

$$\frac{\partial^2 u}{\partial t^2} = c^2 f''(x-ct) = c^2\frac{\partial^2 u}{\partial x^2},$$

which was to be verified. □

8. What is the region R over which you integrate when evaluating the iterated integral
$$\int_1^2 \int_1^x \frac{x}{\sqrt{x^2+y^2}} \, \mathrm{d}y \, \mathrm{d}x?$$
Rewrite this as an iterated integral first with respect to x, then with respect to y. Evaluate this integral. Which order of integration is easier?

***Solution**:* The region $R = \{(x,y) \in \mathbb{R}^2 \mid 1 \leqslant y \leqslant x,\ 1 \leqslant x \leqslant 2\}$ is sketched in Figure 4. Interchanging the order of integration, we obtain that

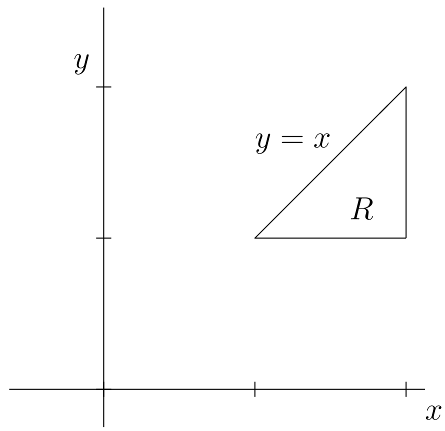

Figure 4: Sketch of Region R

$$\iint_R \frac{x}{\sqrt{x^2+y^2}} \, dxdy = \int_1^2 \int_y^2 \frac{x}{\sqrt{x^2+y^2}} \, dxdy. \tag{25}$$

The iterated integral in (25) is easier to evaluate; in fact,

$$\begin{aligned}
\iint_R \frac{x}{\sqrt{x^2+y^2}} \, dxdy &= \int_1^2 \int_y^2 \frac{x}{\sqrt{x^2+y^2}} \, dxdy \\
&= \int_1^2 \left[\sqrt{x^2+y^2}\right]_y^2 \, dy \\
&= \int_1^2 \left[\sqrt{4+y^2} - \sqrt{2}\, y\right] \, dy.
\end{aligned}$$

We therefore get that

$$\iint_R \frac{x}{\sqrt{x^2+y^2}}\, dxdy \;=\; \int_1^2 \sqrt{4+y^2}\, dy - \sqrt{2}\int_1^2 y\, dy. \tag{26}$$

Evaluating the second integral on the right–hand side of (26) yields

$$\int_1^2 y\, dy = \frac{3}{2}. \tag{27}$$

The first integral on the right–hand side of (26) can be evaluated using the integration formula

$$\int \sqrt{a^2+u^2}\, du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln|u+\sqrt{a^2+u^2}| + C,$$

with $a = 2$, to obtain

$$\int_1^2 \sqrt{4+y^2}\, dy \;=\; \left[\frac{y}{2}\sqrt{4+y^2} + \frac{4}{2}\ln\left|y+\sqrt{4+y^2}\right|\right]_1^2,$$

which evaluates to

$$\int_1^2 \sqrt{4+y^2}\, dy \;=\; 2\sqrt{2} - \frac{\sqrt{5}}{2} + 2\ln\left(\frac{2+\sqrt{8}}{1+\sqrt{5}}\right). \tag{28}$$

Substituting (27) and (28) into (26) we obtain

$$\iint_R \frac{x}{\sqrt{x^2+y^2}}\, dxdy \;=\; \frac{\sqrt{2}}{2} - \frac{\sqrt{5}}{2} + 2\ln\left(\frac{2+\sqrt{8}}{1+\sqrt{5}}\right).$$

□

9. Let $f\colon \mathbb{R}\to\mathbb{R}$ denote a twice–differentiable real valued function and define

$$u(x,y) = f(r) \quad \text{where} \quad r = \sqrt{x^2+y^2} \quad \text{for all} \quad (x,y)\in\mathbb{R}^2.$$

   (a) Define the vector field $F(x,y) = \nabla u(x,y)$. Express F in terms of f' and r.

   ***Solution**:* Compute

$$F(x,y) = \nabla u(x,y) = \frac{\partial u}{\partial x}\,\widehat{\imath} + \frac{\partial u}{\partial y}\,\widehat{\jmath}, \tag{29}$$

where, by the chain rule,

$$\frac{\partial u}{\partial x} = f'(r)\ \frac{\partial r}{\partial x} \tag{30}$$

and

$$\frac{\partial u}{\partial y} = f'(r)\ \frac{\partial r}{\partial y}. \tag{31}$$

To compute $\dfrac{\partial r}{\partial x}$ and $\dfrac{\partial r}{\partial x}$, write

$$r^2 = x^2 + y^2, \tag{32}$$

and differentiate with respect to x on both sides of (32) to obtain

$$2r\frac{\partial r}{\partial x} = 2x,$$

from which we get

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \qquad \text{for } (x, y) \neq (0, 0). \tag{33}$$

Similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{r}, \qquad \text{for } (x, y) \neq (0, 0). \tag{34}$$

Substituting (33) into (30) yields

$$\frac{\partial u}{\partial x} = \frac{f'(r)}{r}\ x. \tag{35}$$

Similarly, substituting (34) into (31) yields

$$\frac{\partial u}{\partial y} = \frac{f'(r)}{r}\ y. \tag{36}$$

Next, substitute (35) and (36) into (29) to obtain

$$F(x, y) = \frac{f'(r)}{r}(x\ \widehat{\imath} + y\ \widehat{\jmath}), \tag{37}$$

$\square$

(b) Express the divergence of the gradient of u, in terms of f', f'' and r.

The expression $\text{div}(\nabla u)$ is called the Laplacian of u, and is denoted by Δu or $\nabla^2 u$.

Solution: From (37) we obtain that

$$P(x,y) = \frac{f'(r)}{r}\, x \quad \text{and} \quad Q(x,y) = \frac{f'(r)}{r}\, y,$$

so that, applying the product rule, the chain rule and quotient rule,

$$\begin{aligned} \frac{\partial P}{\partial x} &= \frac{f'(r)}{r} + x\frac{d}{dr}\left[\frac{f'(r)}{r}\right]\frac{\partial r}{\partial x} \\ &= \frac{f'(r)}{r} + x\, \frac{rf''(r) - f'(r)}{r^2}\, \frac{x}{r}, \end{aligned} \tag{38}$$

where we have also used (33). Simplifying the expression in (38) yields

$$\frac{\partial P}{\partial x} = \frac{f'(r)}{r} + x^2\, \frac{f''(r)}{r^2} - x^2\, \frac{f'(r)}{r^3}. \tag{39}$$

Similar calculations lead to

$$\frac{\partial Q}{\partial y} = \frac{f'(r)}{r} + y^2\, \frac{f''(r)}{r^2} - y^2\, \frac{f'(r)}{r^3}. \tag{40}$$

Adding the results in (39) and (40), we then obtain that

$$\begin{aligned} \text{div}F &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \\ &= 2\, \frac{f'(r)}{r} + r^2\, \frac{f''(r)}{r^2} - r^2\, \frac{f'(r)}{r^3}, \end{aligned} \tag{41}$$

where we have used (32). Simplifying the expression in (41), we get that

$$\text{div}F = f''(r) + \frac{f'(r)}{r}.$$

$\square$

10. Let $f(x,y) = 4x - 7y$ for all $(x,y) \in \mathbb{R}^2$, and $g(x,y) = 2x^2 + y^2$.

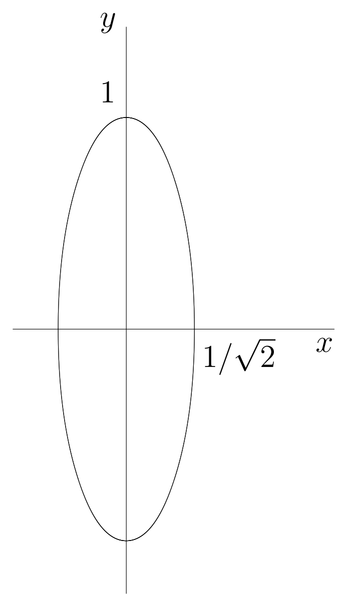

Figure 5: Sketch of ellipse

(a) Sketch the graph of the set $C = g^{-1}(1) = \{(x,y) \in \mathbb{R}^2 \mid g(x,y) = 1\}$.

***Solution**:* The curve C is the graph of the equation

$$\frac{x^2}{1/2} + y^2 = 1,$$

which is sketched in Figure 5. $\square$

(b) Show that at the points where f has an extremum on C, the gradient of f is parallel to the gradient of g.

***Solution**:* Let $\sigma\colon [0, 2\pi] \to \mathbb{R}^2$ denote the C^1 parametrization of C given by

$$\sigma(t) = \left(\frac{\sqrt{2}}{2} \cos t,\ \sin t \right), \qquad \text{for all } t \in [0, 2\pi].$$

We then have that

$$g(\sigma(t)) = 1, \qquad \text{for all } t. \tag{42}$$

Differentiating on both sides of (42) yields that

$$\nabla g(\sigma(t)) \cdot \sigma'(t) = 0, \qquad \text{for all } t,$$

where we have applied the Chain Rule, which shows that $\nabla g(x,y)$ is perpendicular to the tangent vector to C at (x,y).

Next, suppose that $f(\sigma(t))$ has a critical point at t_o. Then, the derivative of $f(\sigma(t))$ at t_o is 0; that is,

$$\nabla f(\sigma(t_o)) \cdot \sigma'(t_0) = 0,$$

where we have applied the Chain Rule. It then follows that $\nabla f(x_o, y_o)$ is perpendicular to the tangent vector to C at a critical point (x_o, y_o). Hence, $\nabla f(x_o, y_o)$ must be parallel to $\nabla g(x_o, y_o)$. $\square$

(c) Find the largest and the smallest value of f on C.

***Solution**:* By the result of part (b), at a critical point, (x, y), of f on C, it must be the case that

$$\nabla g(x, y) = \lambda \nabla f(x, y), \tag{43}$$

for some non–zero real number λ, where

$$\nabla f(x, y) = 4\,\widehat{\imath} - 7\,\widehat{\jmath}, \tag{44}$$

and

$$\nabla g(x, y) = 4x\,\widehat{\imath} + 2y\,\widehat{\jmath}. \tag{45}$$

Substituting (44) and (45) into (43) yields the pair of equations

$$x = \lambda \tag{46}$$

and

$$2y = -7\lambda. \tag{47}$$

Substituting the expressions for x and y in (46) and (47), respectively, into the equation of the ellipse

$$2x^2 + y^2 = 1,$$

yields that

$$\frac{57}{4}\,\lambda^2 = 1,$$

from which we get that

$$\lambda = \pm\frac{2\sqrt{57}}{57}. \tag{48}$$

The values for λ in (48), together with (46) and (47), yield the critical points

$$\left(\frac{2\sqrt{57}}{57}, -\frac{7\sqrt{57}}{57}\right) \quad \text{and} \quad \left(-\frac{2\sqrt{57}}{57}, \frac{7\sqrt{57}}{57}\right). \tag{49}$$

Evaluating the function f at each of the critical points in (49) we obtain that

$$f\left(\frac{2\sqrt{57}}{57}, -\frac{7\sqrt{57}}{57}\right) = \sqrt{57} \quad \text{and} \quad f\left(-\frac{2\sqrt{57}}{57}, \frac{7\sqrt{57}}{57}\right) = -\sqrt{57}.$$

Consequently, the largest value of f on C is $\sqrt{57}$ and the smallest value is $-\sqrt{57}$. □

11. In this problem we consider the line integral $\displaystyle\int_C x\ dx + y\ dy + z\ dz$, where C is any piece–wise C^1 curve in $\mathbb{R}^3$.

(a) If possible, find a C^1 function, f, such that $df = x\ dx + y\ dy + z\ dz$.

***Solution**:* We look for a C^1 function $f\colon \mathbb{R}^3 \to \mathbb{R}$ such that $df = x\ dx + y\ dy + z\ dz$, where

$$df = \frac{\partial f}{\partial x}\ dx + \frac{\partial f}{\partial y}\ dy + \frac{\partial f}{\partial z}\ dz.$$

We then have that

$$\frac{\partial f}{\partial x} = x, \tag{50}$$

$$\frac{\partial f}{\partial y} = y, \tag{51}$$

and

$$\frac{\partial f}{\partial z} = z. \tag{52}$$

It follows from (50) that

$$f(x, y, z) = \frac{x^2}{2} + g(x, y), \qquad \text{for all } (x, y, z) \in \mathbb{R}^3, \tag{53}$$

where $g\colon \mathbb{R}^2 \to \mathbb{R}$ is some C^1 function of two variables.

Taking the partial of f with respect to y in (53) yields

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y}. \tag{54}$$

Thus, comparing (51) and (54),

$$\frac{\partial g}{\partial y} = y. \tag{55}$$

It follows from (55) that

$$g(x, y) = \frac{y^2}{2} + h(z), \quad \text{for } (y, z) \in \mathbb{R}^2, \tag{56}$$

where $h\colon \mathbb{R} \to \mathbb{R}$ is some C^1 function of a single variable.
Next, substitute the expression for g in (56) into the expression for f in (53) to get

$$f(x, y, z) = \frac{x^2}{2} + \frac{y^2}{2} + h(z), \quad \text{for all } (x, y, z) \in \mathbb{R}^3. \tag{57}$$

Taking the partial derivative with respect to z on both sides of (57) yields

$$\frac{\partial f}{\partial z}(x, y, z) = h'(z), \quad \text{for all } (x, y, z) \in \mathbb{R}^3. \tag{58}$$

Comparing the expressions for the partial derivative of f with respect to z in (52) and (58) yields

$$h'(z) = z, \quad \text{for all } z \in \mathbb{R},$$

from which we get that

$$h(z) = \frac{z^2}{2} + c, \quad \text{for all } z \in \mathbb{R}, \tag{59}$$

where c is some constant of integration.
Substituting the expression for h in (59) in the right–hand side of the expression for f in (57) then yields

$$f(x, y, z) = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + c, \quad \text{for all } (x, y, z) \in \mathbb{R}^3, \tag{60}$$

and some constant c. $\square$

(b) Let C be parametrized by a C^1 path connecting the point $P_o(1, -1, -2)$ to the point $P_1(-1, 1, 2)$. Compute the line integral $\int_C x\ dx + y\ dy + z\ dz$.

***Solution**:* It follows from the result on part (a) of this problem that

$$\int_C x\ dx + y\ dy + z\ dz = \int_C df,$$

where f is any of the functions given in (60). Consequently, by the fundamental theorem of Calculus,

$$\int_C x\ dx + y\ dy + z\ dz = f(P_1) - f(P_o). \tag{61}$$

Using the definition of f in (60), we compute

$$f(P_1) = f(-1, 1, 2) = 3 + c$$

and

$$f(P_o) = f(1, -1, -2) = 3 + c.$$

Consequently, in view of (61),

$$\int_C x\ dx + y\ dy + z\ dz = 0.$$

$\square$

(c) Let C denote any simple closed curve in $\mathbb{R}^3$. Evaluate the line integral $\displaystyle\int_C x\ dx + y\ dy + z\ dz.$

***Solution**:* Let $\sigma\colon [0,1] \to \mathbb{R}^3$ be a parametrization of C with $\sigma(0) = P_o$ and $\sigma(1) = P_1$. Then, since C is a closed curve, $P_1 = P_o$; so that,

$$f(P_1) = f(P_o).$$

It then follows from the formula in (61) that

$$\int_C x\ dx + y\ dy + z\ dz = 0.$$

$\square$

12. Let R denote the square, $R = \{(x,y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}$, and ∂R denote the boundary of R oriented in the counterclockwise sense. Evaluate the line integral

$$\int_{\partial R} (y^2 + x^3)\ dx + x^4\ dy.$$

***Solution**:* Apply Green's theorem,

$$\oint_{\partial R} P\ dx + Q\ dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dxdy,$$

where, in this case,

$$P(x,y) = y^2 - x^3 \quad \text{and} \quad Q(x,y) = x^4, \quad \text{for } (x,y) \in \mathbb{R}^2;$$

so that,

$$\int_{\partial R} (y^2 + x^3)\ dx + x^4\ dy \ = \ \iint_R (4x^3 - 2y)\ dxdy, \tag{62}$$

Evaluating the double integral in (62) we obtain that

$$\begin{aligned}
\int_{\partial R} (y^2 + x^3)\ dx + x^4\ dy &= \int_0^1 \int_0^1 (4x^3 - 2y)\ dxdy \\
&= \int_0^1 \left[x^4 - 2xy\right]_0^1\ dy \\
&= \int_0^1 (1 - 2y)\ dy \\
&= \left[y - y^2\right]_0^1 \\
&= 0.
\end{aligned}$$

□