Assignment #19

Due on Wednesday April 16, 2008

Do the following problems

1. Prove that if X and Y are independent random variables,
$$\text{var}(X+Y) = \text{var}(X) + \text{var}(Y).$$
Generalize this result to n independent random variables $X_1, X_2, \ldots, X_n$.

2. Let $X_n \sim \text{Poisson}(n)$, for $n = 1, 2, 3, \ldots$, and define $Z_n = \dfrac{X_n - n}{\sqrt{n}}$ for $n = 1, 2, 3, \ldots$ Use the mgf Convergence Theorem to find the limiting distribution of Z_n.

3. Let X and Y be independent continuous random variables with pdfs f_X and f_Y, respectively. Let $Z = X + Y$ and show that the pdf for Z is given by
$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(u) f_Y(z-u) \, \mathrm{d}u$$
for all $z \in \mathbb{R}$. This is known as the **convolution** of f_X and f_Y.

   Suggestion: To evaluate the double integral defining $P(X + Y \leq z)$, make the change of variables $u = x$ and $v = x + y$. Observe that with this change of variables, the region of integration in the uv–plane becomes: $\{(u, v) \in \mathbb{R}^2 \mid -\infty < u < \infty, -\infty < v < z\}$.

4. Let X and Y be independent $\chi^2(1)$ random variables. Recall that this means that X and Y both have the pdf
$$f(u) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} \dfrac{1}{\sqrt{u}} \, e^{-u/2} & \text{if } u > 0, \\ & \\ 0 & \text{elsewhere.} \end{cases}$$
Let $Z = X + Y$ and use the convolution formula derived in the previous problem to compute the pdf of Z.

   (*Hint:* The distribution of Z is a familiar one).

5. Use the result of the previous problem to compute the moment generating function of a $\chi^2(1)$ random variable.