Assignment #2

Due on Wednesday, February 11, 2009

Read Chapter 2 on *Vector Algebra* in Bressoud (pp. 29–49).

Do the following problems

1. The vectors $v_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, and $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ span a two–dimensional subspace in $\mathbb{R}^3$, in other words, a plane through the origin. Give two unit vectors which are orthogonal to each other, and which also span the plane.

2. Use an appropriate orthogonal projection to compute the shortest distance from the point $P(1, 1, 2)$ to the plane in $\mathbb{R}^3$ whose equation is
$$2x + 3y - z = 6.$$

3. The dual space of of $\mathbb{R}^n$, denoted $(\mathbb{R}^n)^*$, is the vector space of all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}$.

   For a given $w \in \mathbb{R}^n$, define $T_w : \mathbb{R}^n \to \mathbb{R}$ by
$$T_w(v) = w \cdot v \quad \text{for all} \quad v \in \mathbb{R}^n.$$

   Show that T_w is an element of the dual of $\mathbb{R}^n$ for all $w \in \mathbb{R}^n$.

4. Prove that for every linear transformation, $T : \mathbb{R}^n \to \mathbb{R}$, there exists $w \in \mathbb{R}^n$ such that
$$T(v) = w \cdot v \quad \text{for every} \quad v \in \mathbb{R}^n.$$
   (*Hint:* See where T takes the standard basis $\{e_1, e_2, \ldots, e_n\}$ in $\mathbb{R}^n$.)

5. Exercise 19 on page 51 in the text.

6. Exercise 20 on page 51 in the text.

7. Let $u_1, u_2, \ldots, u_n$ be unit vectors in $\mathbb{R}^n$ which are mutually orthogonal; that is,

$$u_i \cdot u_j = 0 \quad \text{for} \quad i \neq j.$$

Prove that the set $\{u_1, u_2, \ldots, u_n\}$ is a basis for $\mathbb{R}^n$, and that, for any $v \in \mathbb{R}^n$,

$$v = \sum_{i=1}^{n} (v \cdot u_i)\ u_i.$$

8. Exercises 22 and 23 on page 51 in the text.

9. In this problem and the next, we derive the vector identity

$$u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$$

for any vectors u, v and w in $\mathbb{R}^3$.

(a) Argue that $u \times (v \times w)$ lies in the span of v and w. Consequently, there exist scalars t and s such that

$$u \times (v \times w) = tv + sw$$

(b) Show that $(u \cdot v)t + (u \cdot w)s = 0$.

10. Let u, v and w be as in the previous problem.

(a) Use the results of the previous problem to conclude that there exists a scalar r such that

$$u \times (v \times w) = r[(u \cdot w)v - (u \cdot v)w].$$

(b) By considering some simple examples, deduce that $r = 1$ in the previous identity