Assignment #3

Due on Wednesday, February 18, 2009

Read Section 7.1 on *Limits,* pp. 171–178, in Bressoud.

Background and Definitions

- *(Open Set)* A subset, U, of $\mathbb{R}^n$ is said to be **open** if for any $x \in U$ there exists a positive number r such that $B_r(x) = \{y \in \mathbb{R}^n \mid \|y - x\| < r\}$ is entirely contained in U.

  (The empty set, $\emptyset$, is considered to be an open set.)

- *(Continuous Function)* Let U denote an open subset of $\mathbb{R}^n$. A function $F\colon U \to \mathbb{R}^m$ is said to be continuous at $x \in U$ if and only if $\lim\limits_{\|y-x\|\to 0} \|F(y) - F(x)\| = 0$.

- *(Image)* If $A \subseteq U$, the *image of* A under the map $F\colon U \to \mathbb{R}^m$, denoted by $F(A)$, is defined as the set $F(A) = \{y \in \mathbb{R}^m \mid y = F(x) \text{ for some } x \in A\}$.

- *(Pre–image)* If $B \subseteq \mathbb{R}^m$, the *pre–image of* B under the map $F\colon U \to \mathbb{R}^m$, denoted by $F^{-1}(B)$, is defined as the set $F^{-1}(B) = \{x \in U \mid F(x) \in B\}$.

  Note that $F^{-1}(B)$ is always defined even if F does not have an inverse map.

Do the following problems

1. Let U_1 and U_2 denote subsets in $\mathbb{R}^n$.

   (a) Show that if U_1 and U_2 are open subsets of $\mathbb{R}^n$, then their intersection

   $$U_1 \cap U_2 = \{y \in \mathbb{R}^n \mid y \in U_1 \text{ and } y \in U_2\}$$

   is also open.

   (b) Show that the set $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; y = 0 \right\}$ is not an open subset of $\mathbb{R}^2$.

2. In Problem 4 of Assignment #2 you proved that every linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}$ must be of the form

   $$T(v) = w \cdot v \quad \text{for every } \ v \in \mathbb{R}^n.$$

   Use this fact together with the Cauchy–Schwarz inequality to prove that T is continuous at every point in $\mathbb{R}^n$.

3. A subset, U, of $\mathbb{R}^n$ is said to be **convex** if given any two points x and y in U, the straight line segment connecting them is entirely contained in U; in symbols,

$$\{x + t(y - x) \in \mathbb{R}^n \mid 0 \leq t \leqslant 1\} \subseteq U$$

   (a) Prove that the ball $B_r(O) = \{x \in \mathbb{R}^n \mid \|x\| < R\}$ is a convex subset of $\mathbb{R}^n$.

   (b) Prove that the "punctured unit disc" in $\mathbb{R}^2$,

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; 0 < x^2 + y^2 < 1 \right\},$$

   is not a convex set.

4. Let x and y denote real numbers.

   (a) Starting with the self–evident inequality: $(|x| - |y|)^2 \geqslant 0$, derive the inequality

$$|xy| \leqslant \frac{1}{2}(x^2 + y^2).$$

   (b) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$$

   Use the inequality derived in the previous part to prove that f is continuous at the origin.

5. Exercise 10 on page 180 in the text.

6. Use the triangle inequality to prove that, for any x and y in $\mathbb{R}^n$,

$$|\|y\| - \|x\|| \leqslant \|y - x\|.$$

   Use this inequality to deduce that the function $f\colon \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x) = \|x\| \quad \text{for all} \quad x \in \mathbb{R}^n$$

   is continuous on $\mathbb{R}^n$.

7. Let $f(x, y)$ and $g(x, y)$ denote two functions defined on a open region, D, in $\mathbb{R}^2$. Prove that the vector field $F\colon D \to \mathbb{R}^2$, defined by

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,$$

is continuous on D if and only f and g are both continuous on D.

8. Let U denote an open subset of $\mathbb{R}^n$ and let $F\colon U \to \mathbb{R}^m$ and $G\colon U \to \mathbb{R}^m$ be two given functions.

   (a) Explain how the sum $F + G$ is defined.

   (b) Prove that if both F and G are continuous on U, then their sum is also continuous.

   (*Suggestion:* The triangle inequality might come in handy.)

9. In each of the following, given the function $F\colon U \to \mathbb{R}^m$ and the set B, compute the pre–image $F^{-1}(B)$.

   (a) $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + y^2 \\ x^2 - y^2 \end{pmatrix}$, and $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.

   (b) $f\colon D' \to \mathbb{R}$,

$$f(x, y) = \frac{1}{\sqrt{1 - x^2 - y^2}}, \quad \text{for} \quad (x, y) \in D'$$

   where $D' = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 < 1\}$ (the punctured unit disc), $B = \{1\}$.

   (c) $f\colon D' \to \mathbb{R}$ is as in part (b), and $B = \{2\}$.

   (d) $f\colon D' \to \mathbb{R}$ is as in part (b), and $B = \{1/2\}$.

10. Compute the image the given sets under the following maps

   (a) $\sigma\colon \mathbb{R} \to \mathbb{R}^2$, $\sigma(t) = (\cos t, \sin t)$ for all $t \in \mathbb{R}$. Compute $\sigma(\mathbb{R})$.

   (b) $f\colon D' \to \mathbb{R}$ and D' are as given in part (b) of the previous problem. Compute $f(D')$.