Assignment #4

Due on Wednesday, February 25, 2009

Read Section 7.1 on *Limits,* pp. 171–178, in Bressoud.

Background and Definitions

- *(Open Set)* A subset, U, of $\mathbb{R}^n$ is said to be **open** if for any $x \in U$ there exists a positive number r such that $B_r(x) = \{y \in \mathbb{R}^n \mid \|y - x\| < r\}$ is entirely contained in U.

  (The empty set, $\emptyset$, is considered to be an open set.)

- *(Continuous Functions 1)* Let U denote an open subset of $\mathbb{R}^n$. A function $F\colon U \to \mathbb{R}^m$ is said to be continuous at $x \in U$ if and only if $\lim\limits_{\|y-x\|\to 0} \|F(y) - F(x)\| = 0$.

  If F is continuous at every point in U, then F is said to be continuous **on** U.

- *(Pre–image)* If $B \subseteq \mathbb{R}^m$, the *pre–image of* B under the map $F\colon U \to \mathbb{R}^m$, denoted by $F^{-1}(B)$, is defined as the set $F^{-1}(B) = \{x \in U \mid F(x) \in B\}$.

  Note that $F^{-1}(B)$ is always defined even if F does not have an inverse map.

- *(Continuous Functions 2)* Let U denote an open subset of $\mathbb{R}^n$. A function $F\colon U \to \mathbb{R}^m$ is continuous on U if and only if, for every open subset V of $\mathbb{R}^m$, the pre–image of V under F, $F^{-1}(V)$ is open in $\mathbb{R}^n$.

- *(Composition of Continuous Functions)* Let U denote an open subset of $\mathbb{R}^n$ and Q an open subset of $\mathbb{R}^m$. Suppose that the maps $F\colon U \to \mathbb{R}^m$ and $G\colon Q \to \mathbb{R}^k$ are continuous on their respective domains and that $F(U) \subseteq Q$. Then, the composition $G \circ F\colon U \to \mathbb{R}^k$ is continuous on U.

Do the following problems

1. Define $G\colon \mathbb{R}^2 \to \mathbb{R}$ by $G(x, y) = xy$ for all $(x, y) \in \mathbb{R}^2$. Prove that G is continuous on $\mathbb{R}^2$; that is, prove that

$$\lim_{(x,y)\to(x_o,y_o)} G(x, y) = G(x_o, y_o) \quad \text{for all} \quad (x_o, y_o) \in \mathbb{R}^2$$

   or

$$\lim_{(x,y)\to(x_o,y_o)} |G(x, y) - G(x_o, y_o)| = 0 \quad \text{for all} \quad (x_o, y_o) \in \mathbb{R}^2$$

2. Let U be an open subset of $\mathbb{R}^2$. Let $f\colon U \to \mathbb{R}$ and $g\colon U \to \mathbb{R}$ be two scalar fields on U, and define $h\colon U \to \mathbb{R}$ by

$$h(x,y) = f(x,y)g(x,y) \quad \text{for all} \quad (x,y) \in U.$$

Prove that if both f and g are continuous on U, then so is h.

Suggestion: Let $F\colon U \to \mathbb{R}^2$ denote the map given by

$$F(x,y) = (f(x,y), g(x,y)) \quad \text{for all} \quad (x,y) \in U,$$

and observe that

$$h = G \circ F,$$

where $G(x,y) = xy$ for all $(x,y) \in \mathbb{R}^2$ is the function defined in the previous problem.

3. Let U denote an open subset of $\mathbb{R}^2$ and let $g\colon U \to \mathbb{R}$ be two scalar fields on U. Assume that $g(x_o, y_o) \neq 0$ for some $(x_o, y_o) \in U$. Prove that if g is continuous at (x_o, y_o), then there exists $\delta > 0$ such that $B_\delta(x_o, y_o) \subseteq U$ and

$$(x,y) \in B_\delta(x_o, y_o) \Rightarrow |g(x,y)| > \frac{|g(x_o,y_o)|}{2}.$$

Suggestion: Consider $\varepsilon = \dfrac{|g(x_o,y_o)|}{2} > 0$.

4. Let U, g and (x_o, y_o) be as in the previous problem. Assume that $g(x_o, y_o) \neq 0$ and that g is continuous at (x_o, y_o). Put

$$h(x,y) = \frac{1}{g(x,y)}.$$

Prove that h is continuous at (x_o, y_o).

Suggestion: Use the result of the previous problem and the Squeeze Theorem.

5. Let U be an open subset of $\mathbb{R}^2$, and $f\colon U \to \mathbb{R}$ and $g\colon U \to \mathbb{R}$ be two scalar fields on U. Use the results of Problems 2 and 4 to make statement regarding the continuity of

$$\frac{f(x,y)}{g(x,y)}$$

at some point $(x_o, y_o) \in U$ and prove the statement.

6. Let U denote an open subset of $\mathbb{R}^n$. Suppose that $f\colon U \to \mathbb{R}$ is a scalar field and $G\colon U \to \mathbb{R}^m$ is vector valued function.

   (a) Explain how the product fG is defined.

   (b) Prove that if both f and G are continuous on U, then the vector valued function fG is also continuous on U.

7. Exercise 3 on page 178 in the text.

8. Exercise 4 on pages 178 and 179 in the text.

9. Exercise 11 on page 180 in the text.

10. Exercise 12 on page 180 in the text.