Assignment #7

Due on Wednesday, April 8, 2009

Read Section 3.1 on *The Calculus of Curves,* pp. 53–65, in Bressoud.

Read Section 5.2 on *Line Integrals,* pp. 113–119, in Bressoud.

Do the following problems

1. Let I denote an open interval in $\mathbb{R}$, and $\sigma\colon I \to \mathbb{R}^n$ be a C^1 path. For fixed $a \in I$, define
$$s(t) = \int_a^t \|\sigma'(\tau)\|\ \mathrm{d}\tau \quad \text{for all} \quad t \in I.$$
Show that s is differentiable and compute $s'(t)$ for all $t \in I$.

2. Let σ and s be as defined in the previous problem. Suppose, in addition, that $\sigma'(t)$ is never the zero vector for all t in I. Show that s is a strictly increasing function of t and that it is, therefore, one–to–one.

3. Let σ and s be as defined in Problem 1. We can re–parameterize σ by using s as a parameter. We therefore obtain $\sigma(s)$, where s is the *arc length* parameter.

   Differentiate the expression
$$\sigma(s(t)) = \sigma(t)$$
   with respect to t using the Chain Rule. Conclude that, if $\sigma'(t)$ is never the zero vector for all t in I, then $\sigma'(s)$ is always a unit vector.

   The vector $\sigma'(s)$ is called the *unit tangent vector* to the path σ.

4. For a and b, positive real numbers, the expression
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
   defines an ellipse in the xy–plane $\mathbb{R}^2$.

   Sketch the ellipse, give a parametrization for it, and set up the integral that yields its arc length.

5. Let $\sigma\colon [0,\pi] \to \mathbb{R}^3$ be defined by $\sigma(t) = t\,\widehat{i} + t\sin t\,\widehat{j} + t\cos t\,\widehat{k}$ for all $t \in [0,\pi]$. Compute the arc length of the curve parametrized by σ.

6. Consider a portion of a helix, C, parametrized by the path

$$\sigma(t) = (\cos t, t, \sin t) \quad \text{for} \quad 0 \leqslant t \leqslant \pi.$$

Let $f(x,y,z) = x^2 + y^2 + z^2$ for all $(x,y,z) \in \mathbb{R}^3$. Evaluate $\displaystyle\int_C f$.

7. Let $f(x,y) = y$ for all $(x,y) \in \mathbb{R}^2$. For each of the following curves, C, in the plane, evaluate $\displaystyle\int_C f$.

   (a) C is the segment along the x axis from $(0,0)$ to $(1,0)$.

   (b) C is the segment along the y axis from $(0,0)$ to $(0,1)$.

   (c) C is the unit circle in $\mathbb{R}^2$.

8. Exercise 10 on page 120 in the text.

9. Exercise 12 on page 120 in the text.

10. Let f be a real valued function which is C^1 in an open interval containing the closed an bounded interval $[a,b]$. Define C to be the portion of the graph of f over $[a,b]$; that is,

$$C = \{(x,y) \in \mathbb{R}^2 \mid y = f(x),\ a \leqslant x \leqslant b\}.$$

   (a) Give a parametrization for C and compute the arc length, $\ell(C)$, of C.

   (b) Compute the arc length along the graph of $y = \ln x$ from $x = 1$ to $x = 2$.
   Note: In order to do part (b), you'll need to remember, or review, everything you learned about evaluating integrals in your single variable Calculus courses.