Exam 1

March 11, 2007

Name: ______________________________

This is a closed book exam. Show all significant work and justify all your answers. Use your own paper and/or the paper provided by the instructor. You have 75 minutes to work on the following 5 problems. Relax.

1. The points $P(1, 0, 0)$, $Q(0, 2, 0)$ and $R(0, 0, 3)$ determine a unique plane in three dimensional Euclidean space, $\mathbb{R}^3$.

   (a) Give the equation of the plane determined by P, Q and R.

   (b) Give the parametric equations of the line through the origin in $\mathbb{R}^3$ which is perpendicular to the plane determined by P, Q and R.

   (c) Find the intersection between the line found in part (b) above and the plane determined by P, Q and R.

   (d) Find the (shortest) distance from the origin in $\mathbb{R}^3$ to the plane determined by P, Q and R.

2. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ denote the scalar field on $\mathbb{R}^2$ defined by

$$f(x, y) = \sqrt{|xy|} \quad \text{for all} \quad (x, y) \in \mathbb{R}^2.$$

   (a) Use the inequality $ab \leqslant \frac{1}{2}(a^2 + b^2)$, for all nonnegative real numbers a and b, and the Squeeze Theorem to prove that f is continuous at $(0, 0)$.

   (b) Show that the partial derivatives of f at $(0, 0)$ exist and compute them.

   (c) Show that f is not differentiable at $(0, 0)$.

3. Let U denote an open subset of $\mathbb{R}^n$, and let $F\colon U \to \mathbb{R}^m$ be a vector field on U.

   (a) State precisely what it means for F to be differentiable at $u \in U$.

   (b) Suppose that a vector field $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. Prove that F is differentiable at every $u \in U$, and compute its derivative map,

$$DF(u)\colon \mathbb{R}^n \to \mathbb{R}^m,$$

   at u, for all $u \in \mathbb{R}^n$.

4. Let U denote an open subset of $\mathbb{R}^n$, and let $f\colon U \to \mathbb{R}$ denote a scalar field on U. Let $\widehat{u}$ denote a unit vector in $\mathbb{R}^n$.

   (a) Prove that if f is differentiable at $v \in U$, then the limit
   $$\lim_{t\to 0}\frac{f(v+t\widehat{u})-f(v)}{t}$$
   exists and is given by the component of the orthogonal projection of the gradient of f at v, $\nabla f(v)$, onto the direction of $\widehat{u}$.

   (b) Denote the limit obtained in part (a) by $D_{\widehat{u}}f(v)$. Give an interpretation for $D_{\widehat{u}}f(v)$.

   (c) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = 3xy - y^2$ for all $(x,y) \in \mathbb{R}^2$. Compute $D_{\widehat{u}}f(2,1)$ where $\widehat{u} = \dfrac{4}{5}\,\widehat{\imath} - \dfrac{3}{5}\,\widehat{\jmath}$.

   (d) Find a direction, $\widehat{u}$, so that $D_{\widehat{u}}f(2,1)$ has the largest possible value.

5. Let $I \subseteq \mathbb{R}$ ba an open interval and $\sigma\colon I \to \mathbb{R}^n$ be a continuous path in $\mathbb{R}^n$.

   (a) State precisely what it means for the path σ to be differentiable at a point $t \in I$.

   (b) Prove that if the path σ is differentiable at $t \in I$, then the function
   $$g\colon I \to \mathbb{R}$$
   defined by
   $$g(t) = \|\sigma(t)\|^2, \quad \text{for all } \ t \in I,$$
   is differentiable at t, and compute $g'(t)$.

   (c) Let g be as defined in part (b) above and suppose that g has a smallest passible value at some point $t_o \in I$. Prove that if g is differentiable at t_o, then $\sigma(t_o)$ and $\sigma'(t_o)$ are orthogonal (or perpendicular) to each other.

   (d) Find the point (or points) along the path in $\mathbb{R}^2$ given by
   $$\sigma(t) = (t, t^2 - 1), \quad \text{for } t \in \mathbb{R},$$
   which are the closest to the origin $(0,0)$ in $\mathbb{R}^2$.
   Suggestion: Use the information given in part (c) above.