Review Problems for Exam 2

1. Consider a wheel of radius a which is rolling on the x–axis in the xy–plane. Suppose that the center of the wheel moves in the positive x–direction and a constant speed v_o. Let P denote a fixed point on the rim of the wheel.

   (a) Give a path $\sigma(t) = (x(t), y(t))$ giving the position of the P at any time t, if P is initially at the point $(0, 2a)$.

   (b) Compute the velocity of P at any time t. When is the velocity of P horizontal? What is the speed of P at those times?

2. Let $f\colon \mathbb{R} \to \mathbb{R}$ denote a twice–differentiable real valued function and define

$$u(x,t) = f(x - ct) \quad \text{for all} \quad (x,t) \in \mathbb{R}^2,$$

   where c is a real constant.

   Show that

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ denote a twice–differentiable real valued function and define

$$u(x,y) = f(r) \quad \text{where} \quad r = \sqrt{x^2 + y^2} \quad \text{for all} \quad (x,y) \in \mathbb{R}^2.$$

   The Laplacian of u, denoted by Δu, is defined to be

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

   Express the Laplacian of u in terms of f', f'' and r.

4. Let $f(x,y) = 4x - 7y$ for all $(x,y) \in \mathbb{R}^2$, and $g(x,y) = 2x^2 + y^2$.

   (a) Sketch the graph of the set $C = g^{-1}(1) = \{(x,y) \in \mathbb{R}^2 \mid g(x,y) = 1\}$.

   (b) Show that at the points where f has an extremum on C, the gradient of f is parallel to the gradient of g.

   (c) Find largest and the smallest value of f on C.

5. Let $C = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1, y \geqslant 0\}$; i.e., C is the upper unit semi–circle. C can be parametrized by

$$\sigma(\tau) = (\tau, \sqrt{1 - \tau^2}) \quad \text{for} \quad -1 \leqslant \tau \leqslant 1.$$

(a) Compute $s(t)$, the arclength along C from $(-1,0)$ to the point $\sigma(t)$, for $0 \leqslant t \leqslant 1$.

(b) Compute $s'(t)$ for $-1 < t < t$ and sketch the graph of s as function of t.

(c) Show that $\cos(\pi - s(t)) = t$ for all $-1 \leqslant t \leqslant 1$, and deduce that

$$\sin(s(t)) = \sqrt{1-t^2} \quad \text{for all} \quad -1 \leqslant t \leqslant 1.$$

6. Let $\omega = 2x \quad \mathrm{d}x + y \quad \mathrm{d}y$ and $\eta = y\ dx - x \quad \mathrm{d}y$ denote differential 1–forms. Compute each of the following $\omega \wedge \ \mathrm{d}\eta$, $\eta \wedge \ \mathrm{d}\omega$ and $\mathrm{d}(\omega \wedge \eta)$.

7. Let C denote the unit circle traversed in the counterclockwise direction. Evaluate the line integral $\displaystyle\int_C x^3 \quad \mathrm{d}y - y^3 \ \mathrm{d}x$.

8. Let $F(x,y) = y\,\widehat{i} - x\,\widehat{j}$ and R be the square in the xy–plane with vertices $(0,0)$, $(2,-1)$, $(3,1)$ and $(1,2)$. Evaluate $\displaystyle\int_{\partial R} F \cdot n \quad \mathrm{d}s$.