Solutions to Review Problems for Exam 2

1. Consider a wheel of radius a which is rolling on the x–axis in the xy–plane. Suppose that the center of the wheel moves in the positive x–direction and a constant speed v_o. Let P denote a fixed point on the rim of the wheel.

   (a) Give a path $\sigma(t) = (x(t), y(t))$ giving the position of the P at any time t, if P is initially at the point $(0, 2a)$.

   Solution: Let $\theta(t)$ denote the angle that the ray from the center

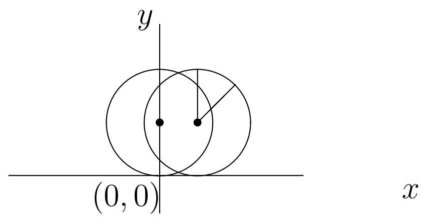

Figure 1: Circle

   of the circle to the point $(x(t), y(t))$ makes with a vertical line through the center. Then, $v_o t = a\theta(t)$; so that $\theta(t) = \dfrac{v_o}{a} t$ and

$$x(t) = v_o t + a\sin(\theta(t))$$

   and

$$y(t) = a + a\cos(\theta(t))$$

□

   (b) Compute the velocity of P at any time t. When is the velocity of P horizontal? What is the speed of P at those times?

   Solution: The velocity vector is

$$\sigma'(t) = (x'(t), y'(t)) = (v_o + a\theta'(t)\cos(\theta(t)), -a\theta'(t)\sin(\theta(t)))$$

   where

$$\theta'(t) = \frac{v_o}{a}.$$

   We then have that

$$\sigma'(t) = (v_o + v_o\cos(\theta(t)), -v_o\sin(\theta(t))).$$

The velocity of P is horizontal when

$$\sin(\theta(t)) = 0,$$

or

$$\theta(t) = n\pi,$$

where n is an integer; and when

$$\cos(\theta(t)) \neq -1.$$

We then get that the velocity of P is horizontal when

$$\theta(t) = 2k\pi$$

where k is an integer.
The speed at the points where the velocity if horizontal is then equal to $2v_o$. $\square$

2. Let $f\colon \mathbb{R} \to \mathbb{R}$ denote a twice–differentiable real valued function and define

$$u(x,t) = f(x - ct) \quad \text{for all} \quad (x,t) \in \mathbb{R}^2,$$

where c is a real constant.

Show that

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

***Solution**:* Use the Chain Rule to compute

$$\frac{\partial u}{\partial t} = f'(x - ct) \cdot \frac{\partial}{\partial t}(x - ct) = -c\ f'(x - ct),$$

and

$$\frac{\partial^2 u}{\partial t^2} = c\ f''(x - ct) \cdot \frac{\partial}{\partial t}(x - ct) = c^2\ f''(x - ct).$$

Similarly,

$$\frac{\partial^2 u}{\partial x^2} = f''(x - ct)$$

since $\dfrac{\partial}{\partial x}(x - ct) = 1$. Hence,

$$\frac{\partial^2 u}{\partial t^2} = c^2\ f''(x - ct) = c^2 \frac{\partial^2 u}{\partial x^2}.$$

$\square$

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ denote a twice–differentiable real valued function and define

$$u(x, y) = f(r) \quad \text{where} \quad r = \sqrt{x^2 + y^2} \quad \text{for all} \quad (x, y) \in \mathbb{R}^2.$$

The Laplacian of u, denoted by Δu, is defined to be

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Express the Laplacian of u in terms of f', f'' and r.

***Solution**:* First note that $r^2 = x^2 + y^2$, from which we get that

$$2r\frac{\partial r}{\partial x} = 2x,$$

or

$$\frac{\partial r}{\partial x} = \frac{x}{r}.$$

Similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{r}.$$

Next, use the Chain Rule to compute

$$\frac{\partial u}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x} = f'(r)\frac{x}{r}.$$

Differentiating with respect to x again, using the Chain, Product and Quotient rules,

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(x\frac{f'(r)}{r}\right) \\
&= \frac{f'(r)}{r} + x\frac{\partial}{\partial x}\left(\frac{f'(r)}{r}\right) \\
&= \frac{f'(r)}{r} + x\frac{rf''(r)\frac{x}{r} - f'(r)\frac{x}{r}}{r^2} \\
&= \frac{f'(r)}{r} + \frac{x^2}{r^2}f''(r) - \frac{x^2}{r^3}f'(r)
\end{aligned}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r} + \frac{y^2}{r^2}f''(r) - \frac{y^2}{r^3}f'(r).$$

Hence

$$\begin{aligned}
\Delta u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\
&= 2\frac{f'(r)}{r} + \frac{x^2+y^2}{r^2} f''(r) - \frac{x^2+y^2}{r^3} f'(r) \\
&= 2\frac{f'(r)}{r} + \frac{r^2}{r^2} f''(r) - \frac{r^2}{r^3} f'(r) \\
&= 2\frac{f'(r)}{r} + f''(r) - \frac{1}{r} f'(r) \\
&= f''(r) + \frac{1}{r} f'(r).
\end{aligned}$$

□

4. Let $f(x,y) = 4x - 7y$ for all $(x,y) \in \mathbb{R}^2$, and $g(x,y) = 2x^2 + y^2$.

   (a) Sketch the graph of the set $C = g^{-1}(1) = \{(x,y) \in \mathbb{R}^2 \mid g(x,y) = 1\}$.

   Solution: The curve C is given by the set of points (x,y) in $\mathbb{R}^2$ such that

   $$2x^2 + y^2 = 1,$$

   or

   $$\frac{x^2}{1/2} + y^2 = 1.$$

   That is, C is an ellipse with minor vertices $\pm 1/\sqrt{2}$ and major vertices ± 1. The sketch is shown in Figure 2. □

   (b) Show that at the points where f has an extremum on C, the gradient of f is parallel to the gradient of g.

   Solution: Let $\sigma(t)$ be a parametrization of the ellipse. We want to find a value of t for which the function $f(\sigma(t))$ is the largest. Thus, we first look for critical points of this function. By the Chain Rule,

   $$\frac{d}{dt}\left(f(\sigma(t))\right) = \nabla f(\sigma(t)) \cdot \sigma'(t).$$

   Thus, t is a critical point if the tangent vector $\sigma'(t)$ is perpendicular to $\nabla f(x,y) = 4\widehat{i} - 7\widehat{j}$.

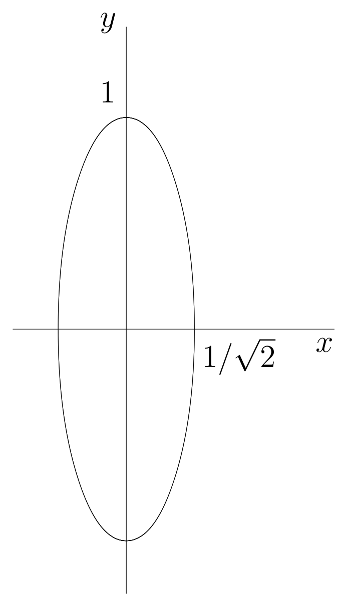

Figure 2: Sketch of ellipse

On the other hand, from

$$g(\sigma(t)) = 1 \quad \text{for all t}$$

we get that

$$\nabla g(\sigma(t)) \cdot \sigma'(t) = 0$$

so that $\sigma'(t)$ is also perpendicular to $\nabla g(x, y) = 4x\widehat{i} + 2y\widehat{j}$. Hence, ∇f and ∇g must be parallel at a critical points; that is, there must be a constant $\lambda \neq 0$ such that

$$\nabla g(x, y) = \lambda \nabla f(x, y). \tag{1}$$

$\square$

(c) Find largest and the smallest value of f on C.

Solution: To find the critical points of f on C we use the condition (1) derived in the previous part, or

$$4x\widehat{i} + 2y\widehat{j} = 4\lambda\widehat{i} - 7\lambda\widehat{j}.$$

We then get that

$$4x = 4\lambda$$

and

$$2y = -7\lambda.$$

In other words, a critical point (x, y) must lie in the line

$$2y = -7x.$$

Next, we find the intersection of this line with the ellipse.
Solving for y and substituting into the equation of the ellipse we get that

$$2x^2 + \left(\frac{-7x}{2}\right)^2 = 1$$

or

$$2x^2 + \frac{49}{4}x^2 = 1$$

or

$$\frac{57}{4}x^2 = 1$$

or

$$x^2 = \frac{4}{57}$$

from which we get that

$$x = \pm\frac{2}{\sqrt{57}}.$$

We therefore get the critical points

$$\left(\frac{2}{\sqrt{57}}, -\frac{7}{\sqrt{57}}\right) \quad \text{and} \quad \left(-\frac{2}{\sqrt{57}}, \frac{7}{\sqrt{57}}\right).$$

Evaluating f at each of these points we find that

$$f\left(\frac{2}{\sqrt{57}}, -\frac{7}{\sqrt{57}}\right) = \frac{8}{\sqrt{57}} + \frac{49}{\sqrt{57}} = \sqrt{57}$$

and

$$f\left(-\frac{2}{\sqrt{57}}, \frac{7}{\sqrt{57}}\right) = -\frac{8}{\sqrt{57}} - \frac{49}{\sqrt{57}} = -\sqrt{57}.$$

Thus, f is the largest at $\left(\frac{2}{\sqrt{57}}, -\frac{7}{\sqrt{57}}\right)$ and the smallest at $\left(-\frac{2}{\sqrt{57}}, \frac{7}{\sqrt{57}}\right)$. The largest value of f on C is then $\sqrt{57}$, and its smallest value on C is $-\sqrt{57}$. $\square$

5. Let $C = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1, y \geqslant 0\}$; i.e., C is the upper unit semi–circle. C can be parametrized by

$$\sigma(\tau) = (\tau, \sqrt{1-\tau^2}) \quad \text{for} \quad -1 \leqslant \tau \leqslant 1.$$

(a) Compute $s(t)$, the arclength along C from $(-1,0)$ to the point $\sigma(t)$, for $0 \leqslant t \leqslant 1$.

Solution: Compute $\sigma'(\tau) = \left(1, -\dfrac{\tau}{\sqrt{1-\tau^2}}\right)$. for all $\tau \in (-1,1)$.
Then,

$$\|\sigma'(\tau)\| = \sqrt{1 + \frac{\tau^2}{1-\tau^2}} = \frac{1}{\sqrt{1-\tau^2}}.$$

It then follows that

$$s(t) = \int_{-1}^{t} \frac{1}{\sqrt{1-\tau^2}} \, \mathrm{d}\tau \quad \text{for} \ -1 \leqslant t \leqslant 1.$$

□

(b) Compute $s'(t)$ for $-1 < t < t$ and sketch the graph of s as function of t.

Solution: By the Fundamental Theorem of Calculus,

$$s'(t) = \frac{1}{\sqrt{1-t^2}} \quad \text{for} \ -1 < t < 1.$$

Note then that $s'(t) > 0$ for all $t \in (-1,1)$ and therefore s is strictly increasing on $(-1,1)$.
Next, compute the derivative of $s'(t)$ to get the second derivative of $s(t)$:

$$s''(t) = \frac{t}{(1-t^2)^{3/2}} \quad \text{for} \ -1 < t < 1.$$

It then follows that $s''(t) < 0$ for $-1 < t < 0$ and $s''(t) > 0$ for $0 < t < 1$. Thus, the graph of $s = s(t)$ is concave down on $(-1,0)$ and concave up on $(0,1)$.
Finally, observe that $s(-1) = 0$, $s(0) = \pi/2$ (the arc–length along a quarter of the unit circle), and $s(1) = \pi$ (the arc–length along a semi–circle of unit radius). We can then sketch the graph of $s = s(t)$ as shown in Figure 3. □

(c) Show that $\cos(\pi - s(t)) = t$ for all $-1 \leqslant t \leqslant 1$, and deduce that

$$\sin(s(t)) = \sqrt{1-t^2} \quad \text{for all} \quad -1 \leqslant t \leqslant 1.$$

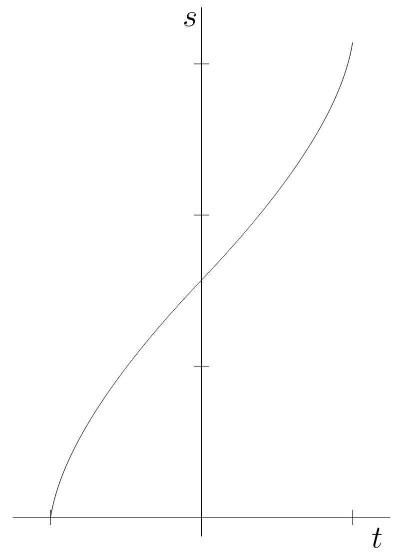

Figure 3: Sketch of $s = s(t)$

***Solution**:* Figure 4 shows the upper unit semicircle and a point $\sigma(t)$ on it. Putting $\theta(t) = \pi - s(t)$, then $\theta(t)$ is the angle, in radians, that the ray from the origin to $\sigma(t)$ makes with the positive x–axis. It then follows that

$$\cos(\theta(t)) = t$$

and

$$\sin(\theta(t)) = \sqrt{1 - t^2}.$$

Since

$$\sin(\theta(t)) = \sin(\pi - s(t)) = \sin(s(t),$$

the result follows. $\square$

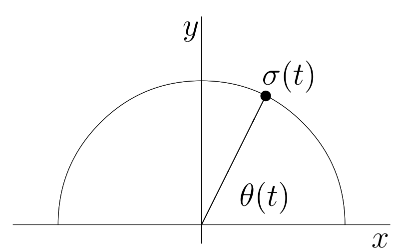

Figure 4: Sketch of Semi–circle

6. Let $\omega = 2x \ \ \mathrm{d}x + y \ \ \mathrm{d}y$ and $\eta = y\ dx - x \ \ \mathrm{d}y$ denote differential 1–forms. Compute each of the following $\omega \wedge \ \mathrm{d}\eta$, $\eta \wedge \ \mathrm{d}\omega$ and $\mathrm{d}(\omega \wedge \eta)$.

   ***Solution**:* Compute

$$\mathrm{d}\omega = \mathrm{d}(2x \ \ \mathrm{d}x + y \ \ \mathrm{d}y) = 2 \ \mathrm{d}x \wedge \ \mathrm{d}x + \ \mathrm{d}y \wedge \ \mathrm{d}y = 0,$$

$$\mathrm{d}\eta = \mathrm{d}(y\ dx - x \ \ \mathrm{d}y) = \mathrm{d}y \wedge \ \mathrm{d}x - \ \mathrm{d}x \wedge \ \mathrm{d}y = -2 \ \mathrm{d}x \wedge \ \mathrm{d}y.$$

   Then

$$\omega \wedge \mathrm{d}\eta = (2x \ \mathrm{d}x + y \ \mathrm{d}y) \wedge (-2 \,\mathrm{d}x \wedge \mathrm{d}y) = -4x \,\mathrm{d}x \wedge \mathrm{d}x \wedge \mathrm{d}y - 2y \,\mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}y = 0,$$

   since $\mathrm{d}x \wedge \ \mathrm{d}x = 0$ and $\mathrm{d}y \wedge \ \mathrm{d}x \wedge \ \mathrm{d}y = - \ \mathrm{d}x \wedge \ \mathrm{d}y \wedge \ \mathrm{d}y = 0$, and

$$\eta \wedge \ \mathrm{d}\omega = \eta \wedge 0 = 0.$$

   Finally,

$$\begin{aligned}
\omega \wedge \eta &= (2x \ \ \mathrm{d}x + y \ \ \mathrm{d}y) \wedge (y\ dx - x \ \ \mathrm{d}y) \\
&= 2xy \ \ \mathrm{d}x \wedge \ \mathrm{d}x - 2x^2 \ \mathrm{d}x \wedge \ \mathrm{d}y + y^2 \ \mathrm{d}y \wedge \ \mathrm{d}x - xy \ \ \mathrm{d}y \wedge \ \mathrm{d}y \\
&= -(2x^2 + y^2) \ \mathrm{d}x \wedge \ \mathrm{d}y;
\end{aligned}$$

   so that

$$\mathrm{d}(\omega \wedge \eta) = -(4x \ \ \mathrm{d}x + 2y \ \ \mathrm{d}y) \ \mathrm{d}x \wedge \ \mathrm{d}y = 0.$$

   □

7. Let C denote the unit circle traversed in the counterclockwise direction. Evaluate the line integral $\displaystyle\int_C x^3 \ \ \mathrm{d}y - y^3 \ \mathrm{d}x$.

   ***Solution**:* Observe that $\displaystyle\int_C x^3 \ \ \mathrm{d}y - y^3 \ \mathrm{d}x$ is the flux of the vector field $F(x, y) = x^3\widehat{i} + y^3\widehat{j}$, so that, by the divergence form of the Fundamental Theorem of Calculus in $\mathbb{R}^2$,

$$\int_C x^3 \ \ \mathrm{d}y - y^3 \ \mathrm{d}x = \int_D \mathrm{div}F \ \ \mathrm{d}x \ \mathrm{d}y,$$

   where D is the unit disc in $\mathbb{R}^2$ centered at the origin, and

$$\mathrm{div}F = 3x^2 + 3y^2 = 3(x^2 + y^2).$$

Using polar coordinates we then get that

$$\begin{aligned}\int_C x^3 \ \mathrm{d}y - y^3 \ \mathrm{d}x &= \int_0^{2\pi}\int_0^1 3r^2 \ r \ \mathrm{d}r \ \mathrm{d}\theta \\ &= 6\pi \int_0^1 r^3 \ \mathrm{d}r \\ &= \frac{3\pi}{2}.\end{aligned}$$

$\square$

8. Let $F(x, y) = y\,\widehat{i} - x\,\widehat{j}$ and R be the square in the xy–plane with vertices $(0, 0)$, $(2, -1)$, $(3, 1)$ and $(1, 2)$. Evaluate $\displaystyle\int_{\partial R} F \cdot n \ \mathrm{d}s$.

***Solution**:* Observe that the divergence of F is

$$\mathrm{div}F = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-x) = 0$$

for all $(x, y) \in \mathbb{R}^2$, so that, by the divergence form of the Fundamental Theorem of Calculus in $\mathbb{R}^2$,

$$\int_{\partial R} F \cdot n \ \mathrm{d}s = \int_R \mathrm{div}F \ \mathrm{d}x \ \mathrm{d}y = 0.$$

$\square$