Topics for Exam 2

1. **Differentiability**

   1.1 Sufficient conditions for differentiability

   1.2 The Jacobain matrix

   1.3 Differentiability of Compositions: The Chain Rule

2. **Path Integrals**

   2.1 C^1 curves and parametrizations

   2.2 Arclength

   2.3 Definition of the path integral

3. **Line Integrals**

   3.1 Definition of the line integral

   3.2 Gradient fields

   3.3 Flux across a closed curve in the plane

4. **Differential forms**

   4.1 Directed line segments and oriented triangles

   4.2 Differential forms (0–forms, 1–forms and 2–forms)

   4.3 Calculus of differential forms

   4.4 Evaluating 2–forms: the double integral

5. **The Fundamental Theorem of Calculus**

   5.1 Stokes' Theorem

   5.2 Green's Theorem

Relevant chapters and sections in the text: Section 7.4 on *The Derivative*, Section 7.6 on *The Chain Rule*, Section 3.1 on *The Calculus of Curves*, Section 5.2 on *Line Integrals*, Chapter 4 on *Differential Forms* and Section 5.4 on *Multiple Integrals*.

Relevant chapters in the online class notes: Sections 4.5 and 4.6, Chapter 5 up to Section 5.8.

Important Concepts: C^1 curves, simple curves, parametrizations, arclength, path integral, line integral, flux, differential forms and double integrals

Important Skills: Know how to apply the Chain Rule, know how to evaluate path integrals, know how to evaluate line integrals, know how to evaluate differential forms, know how to evaluate double integrals, and know how to apply the generalized Fundamental theorem of Calculus, or Stokes' theorem.

Some Formulas

1. **Jacobian Matrix of a C^1 Function**

   The Jacobian matrix of a function $\Phi\colon D \to \mathbf{R}^2$ defined on an open subset, D, of $\mathbf{R}^2$ by
   $$\Phi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} u \\ v \end{pmatrix} \in D,$$
   where x and y are C^1 scalar fields on D, is given by
   $$D\Phi(u,v) = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix},$$
   where the partial derivatives are evaluated at (u,v) in D.

2. **Jacobian Determinant of a C^1 Function**

   The Jacobian determinant, or simply the Jacobian, of a C^1 function $\Phi\colon D \to \mathbf{R}^2$ is the determinant of the Jacobian matrix $D\Phi(u,v)$. We denote it by $\dfrac{\partial(x,y)}{\partial(u,v)}$.

3. **Tangent Line Approximation to a C^1 Path**

   The tangent line approximation to a C^1 path $\sigma\colon [a,b] \to \mathbf{R}^n$ at $\sigma(t_o)$, for some $t_o \in (a,b)$, is the straight line given by
   $$L(t) = \sigma(t_o) + (t - t_o)\sigma'(t_o) \quad \text{for all} \quad t \in \mathbf{R}$$

4. **Arc Length**

   Let $\sigma\colon [a,b] \to \mathbf{R}^n$ be a C^1 parametrization of a curve C. The arc length of C is given by
   $$\ell(C) = \int_a^b \|\sigma'(t)\| \ \mathrm{d}t.$$

5. **Path Integral**

   Let $f\colon U \to \mathbf{R}$ be a continuous scalar field defined on some open subset of $\mathbf{R}^n$. Suppose there is a C^1 curve C contained in U. Then the integral of f over C is given by

$$\int_C f \ \ \mathrm{d}s = \int_a^b f(\sigma(t))\|\sigma'(t)\| \ \ \mathrm{d}t,$$

   for any C^1 parametrization, $\sigma\colon [a,b] \to \mathbf{R}^n$ of the curve C.

6. **Line Integral**

   Let $F\colon U \to \mathbf{R}^n$ denote a continuous vector field defined on some open subset, U, of $\mathbf{R}^n$. Suppose there is a C^1 curve, C, contained in U. Then, the line integral of F over C is given by

$$\int_C F \cdot T \ \ \mathrm{d}s = \int_a^b F(\sigma(t)) \cdot \sigma'(t) \ \ \mathrm{d}t,$$

   for any C^1 parametrization, $\sigma\colon [a,b] \to \mathbf{R}^n$, of the curve C. Here T denotes the tangent unit vector to the curve, and it is given by

$$T(t) = \frac{1}{\|\sigma'(t)\|}\sigma'(t) \quad \text{for all} \ \ t \in (a,b).$$

   If $F = P\,\widehat{i} + Q\,\widehat{j} + R\,\widehat{k}$, where P Q and R are C^1 scalar fields defined on U,

$$\int_C F \cdot T \ \ \mathrm{d}s = \int_C P \ \ \mathrm{d}x + Q \ \ \mathrm{d}y + R \ \ \mathrm{d}z.$$

   The expression $P \ \ \mathrm{d}x + Q \ \ \mathrm{d}y + R \ \ \mathrm{d}z$ is called a differential 1–form.

7. **Flux**

   Let $F = P\,\widehat{i} + Q\,\widehat{j}$, where P and Q are continuous scalar fields defined on an open subset, U, of $\mathbf{R}^2$. Suppose there is a C^1 simple closed curve C contained in U. Then the flux of F across C is given by

$$\int_C F \cdot \widehat{n} \ \ \mathrm{d}s = \int_C P \ \mathrm{d}y - Q \ \mathrm{d}x.$$

   Here, $\widehat{n}$ denotes a unit vector perpendicular to C and pointing to the outside of C.

8. **Green's Theorem.**

The Fundamental Theorem of Calculus,

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega,$$

takes the following form in two–dimensional Euclidean space:

Let R denote a region in $\mathbf{R}^2$ bounded by a simple closed curve, $\partial\mathbf{R}$, made up of a finite number of C^1 paths traversed in the counterclockwise sense. Let P and Q denote two C^1 scalar fields defined on some open set containing R and its boundary, ∂R. Then,

$$\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\ \mathrm{d}y = \int_{\partial R} P\ \mathrm{d}x + Q\ \mathrm{d}y.$$