Assignment #12

Due on Wednesday, March 4, 2009

Read Section 4.1 on *Inner Products and Norms* in Messer (pp. 135–140).
Read Section 4.2 on *Geometry in Euclidean Spaces* in Messer (pp. 143–147).
Read Section 4.3 on *The Cauchy–Schwarz Inequality* in Messer (pp. 149–153).

Read Section 4.4 on *Orthogonality* in Messer (pp. 155–161).

Background and Definitions

- *(Transpose of a vector).* Given a vector $v = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ in $\mathbb{R}^n$, the **transpose** of v, denoted by v^T, is the row vector
$$v^T = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}.$$

- *(Row–Column Product).* Given a row–vector, R, of dimension n and a column–vector, C, also of dimension n, we define the product RC as follows:

  Write $R = [x_1 \quad x_2 \quad \cdots \quad x_n]$ and $C = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$; then,
$$RC = [x_1 \quad x_2 \quad \cdots \quad x_n] \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

- *(Euclidean inner product).* Given vectors $v = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $w = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ in $\mathbb{R}^n$, the *Euclidean inner product* of v and w, denoted by $\langle v, w\rangle$, is the real number (or scalar) obtained by follows
$$\langle v, w\rangle = v^T w = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

- *(Orthogonality).* Two vectors v and w in $\mathbb{R}^n$ are said to be **orthogonal** if $\langle v, w\rangle = 0$.

- *(Euclidean norm).* Given a vector $v = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ in $\mathbb{R}^n$, its **Euclidean norm**, denoted by $\|v\|$, is defined by

$$\|v\| = \sqrt{\langle v, v\rangle} = \sqrt{x^2 + x_2^2 + \cdots + x_n^2}.$$

- *(Unit vectors in $\mathbb{R}^n$).* A vector $u \in \mathbb{R}^n$ is said to be a **unit vector** if $\|u\| = 1$.

Do the following problems

1. The vectors $v_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, and $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ span a two–dimensional subspace in $\mathbb{R}^3$, in other words, a plane through the origin. Give two unit vectors which are orthogonal to each other, and which also span the plane.

2. Let $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; 3x - 2y + z = 0 \right\}$. Find a non–zero vector v in $\mathbb{R}^3$ which is orthogonal to every vector in W; that is, $v \neq \mathbf{0}$ and

$$\langle v, w\rangle = 0 \quad \text{for all} \quad w \in W.$$

3. Let $u_1, u_2, \ldots, u_n$ be unit vectors in $\mathbb{R}^n$ which are mutually orthogonal; that is,

$$\langle u_i, u_j\rangle = 0 \quad \text{for} \quad i \neq j.$$

Prove that the set $\{u_1, u_2, \ldots, u_n\}$ is a basis for $\mathbb{R}^n$, and that, for any $v \in \mathbb{R}^n$,

$$v = \sum_{i=1}^{n} \langle v, u_i\rangle \; u_i.$$

4. The Euclidean inner product of two vectors in $\mathbb{R}^n$ is symmetric, bi–linear and positive definite; that is, for vectors v, v_1, v_2 and w in $\mathbb{R}^n$,

   (i) $\langle v, w\rangle = \langle w, v\rangle$,

   (ii) $\langle c_1 v_1 + c_2 v_2, w\rangle = c_1\langle v_1, w\rangle + c_2\langle v_2, w\rangle$, and

   (iii) $\langle v, v\rangle \geqslant 0$ for all $v \in \mathbb{R}^n$ and $\langle v, v\rangle = 0$ if and only if v is the zero vector.

   Use these properties of the the inner product in $\mathbb{R}^n$ to derive the following properties of the norm $\|\cdot\|$ in $\mathbb{R}^n$:

   (a) $\|v\| \geqslant 0$ for all $v \in \mathbb{R}^n$ and $\|v\| = 0$ if and only if $v = \mathbf{0}$.

   (b) For a scalar c, $\|cv\| = |c|\|v\|$.

5. The Cauchy-Schwarz inequality for any vectors v and w in $\mathbb{R}^n$ states that

   $$|\langle v, w\rangle| \leqslant \|v\|\|w\|.$$

   Use this inequality to derive the triangle inequality: For any vectors v and w in $\mathbb{R}^n$,

   $$\|v + w\| \leqslant \|v\| + \|w\|.$$

   (*Suggestion:* Start with the expression $\|v + w\|^2$ and use the properties of the inner product to simplify it.)