Assignment #20

Due on Wednesday, April 15, 2009

Read Section 6.1 on *Linear Functions* in Messer (pp. 212–216).

Read Section 6.3 on *Matrix of a Linear Function* in Messer (pp. 226–231).

Read Section 6.2 on *Compositions and Inverses* in Messer (pp. 218–223).

Do the following problems

1. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a function satisfying

$$f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \; f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \quad \text{and} \quad f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

   (a) Show that f cannot be linear.

   (b) What would $f\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ be if f was a linear function?

2. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear function satisfying

$$T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}.$$

   (a) Find the matrix representation for T relative to the standard bases in $\mathbb{R}^2$ and $\mathbb{R}^3$.

   (b) Give formula for computing $T\begin{pmatrix} x \\ y \end{pmatrix}$ for any $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$.

   (c) Compute $T\begin{pmatrix} 4 \\ 7 \end{pmatrix}$.

3. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the linear transformation defined in Problem 2.

   (a) Determine the image, $\mathcal{I}_T = \{w \in \mathbb{R}^3 \mid w = T(v) \text{ for some } v \in \mathbb{R}^2\}$, of T.

   (b) Find a basis for $\mathcal{I}_T$ and compute $\dim(\mathcal{I}_T)$.

4. The projection $P_u\colon \mathbb{R}^3 \to \mathbb{R}^3$ onto the direction of the unit vector u in $\mathbb{R}^3$ is given by
$$P_u(v) = \langle v, u\rangle\, u \quad \text{for all } v \in \mathbb{R}^3,$$
where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^3$. We proved in class that P_u is a linear function.

   (a) For $u = \dfrac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, give the matrix representation for P_u relative to the standard basis in $\mathbb{R}^3$.

   (b) For u as defined in the previous part, determine the null space,
$$\mathcal{N}_{P_u} = \{v \in \mathbb{R}^3 \mid P_u(v) = \mathbf{0}\},$$
of P_u.

   (c) Find a basis for $\mathcal{N}_{P_u}$ and compute $\dim(\mathcal{N}_{P_u})$.

5. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ and $R\colon \mathbb{R}^m \to \mathbb{R}^k$ denote two linear functions. The composition of R and T, denoted by $R \circ T$, is the function $R \circ T\colon \mathbb{R}^n \to \mathbb{R}^k$ defined by
$$R \circ T(v) = R(T(v)) \quad \text{for all } \ v \in \mathbb{R}^n.$$

   (a) Prove that the composition $R \circ T$ is a linear function from $\mathbb{R}^n$ to $\mathbb{R}^k$.

   (b) Show that $\mathcal{N}_T \subseteq \mathcal{N}_{R\circ T}$.

   (c) Show that $\mathcal{I}_{R\circ T} \subseteq \mathcal{I}_R$.