Assignment #21

Due on Monday, April 20, 2009

Read Section 6.1 on *Linear Functions* in Messer (pp. 212–216).

Read Section 6.3 on *Matrix of a Linear Function* in Messer (pp. 226–231).

Read Section 6.2 on *Compositions and Inverses* in Messer (pp. 218–223).

Do the following problems

1. Given two vector–valued functions, T and R, from $\mathbb{R}^n$ to $\mathbb{R}^m$, we can define the sum, $T+R$, of T and R by

$$(T+R)(v) = T(v) + R(v) \quad \text{for all} \quad v \in \mathbb{R}^n.$$

   (a) Verify that, if both T and R are linear, then so is $T+R$.

   (b) Explain how to define the scalar multiple $aT\colon \mathbb{R}^n \to \mathbb{R}^m$ of a vector valued function, $T\colon \mathbb{R}^n \to \mathbb{R}^m$, where a is a scalar and verify that if T is linear then so is aT.

2. The **identity** function, $I\colon \mathbb{R}^n \to \mathbb{R}^n$, is defined by

$$I(v) = v \quad \text{for all} \quad v \in \mathbb{R}^n.$$

   (a) Verify that $I\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation.

   (b) Give the matrix representation of I relative to the standard basis in $\mathbb{R}^n$.

   (c) Compute the null space, $\mathcal{N}_I$, and image, $\mathcal{I}_I$, of I.

3. The **zero** function, $O\colon \mathbb{R}^n \to \mathbb{R}^m$, is defined by

$$O(v) = \mathbf{0} \quad \text{for all} \quad v \in \mathbb{R}^n.$$

   (a) Verify that $O\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

   (b) Give the matrix representation of O relative to the standard bases in $\mathbb{R}^n$ and $\mathbb{R}^m$.

   (c) Compute the null space, $\mathcal{N}_O$, and image, $\mathcal{I}_O$, of O.

4. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ denote a linear function and let $M_T \in \mathbb{M}(m,n)$ be its matrix representation with respect to the standard bases in $\mathbb{R}^n$ and $\mathbb{R}^m$.

   (a) Prove that the null space of T, $\mathcal{N}_T$, is the null space of the matrix M_T.

   (b) Prove that the image of T, $\mathcal{I}_T$, is the span of the columns of the matrix M_T.

5. If $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a function, we can define the **iterates**, T^k, of T, where k is a positive integer, as follows:
$$T^2 = T \circ T;$$
   That is, T is the composition of T with itself. Next, define
$$T^3 = T^2 \circ T$$
   and so on. More precisely, once we have defined T^{k-1} for $k > 1$, we can define T^k by
$$T^k = T^{k-1} \circ T.$$

   (a) Prove that if T is a linear function from $\mathbb{R}^n$ to $\mathbb{R}^n$, then so are the functions T^k for $k = 1, 2, \ldots$

   (b) Prove that T^m and T^k commute with each other; that is,
$$T^m \circ T^k = T^k \circ T^m,$$
   where k and m are positive integers.

   (c) Given $v \in \mathbb{R}^n$, prove that the set
$$\{v, T(v), T^2(v), \ldots, T^n(v)\}$$
   is linearly dependent.