Assignment #23

Due on Monday, April 27, 2009

Read Section 6.1 on *Linear Functions* in Messer (pp. 212–216).

Read Section 6.3 on *Matrix of a Linear Function* in Messer (pp. 226–231).

Read Section 6.2 on *Compositions and Inverses* in Messer (pp. 218–223).

Read Section 7.2 on *Definition* of the Determinant in Messer (pp. 273–276).

Background and Definitions

Orthogonal Transformation. An $n \times n$ matrix, A, is said to be **orthogonal** if

$$A^T A = I,$$

where I denotes the identity matrix in $\mathbb{M}(n,n)$.

A linear transformation, $R\colon \mathbb{R}^n \to \mathbb{R}^n$ is said to be orthogonal, if its matrix representation with respect to the standard basis in $\mathbb{R}^n$, M_T, is an orthogonal matrix.

Determinant of a 2×2 matrix. The determinant of the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is define to be

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Geometrically, the absolute value of the determinant of A gives the area of the parallelogram determined by the columns of A.

Do the following problems

1. Let R_1 and R_2 denote two orthogonal transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$. Prove that the composition $R_2 \circ R_1$ is also an orthogonal transformation.

2. Let T_1 and T_2 denote two linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$. Prove that if the composition $T_2 \circ T_1$ is singular, then either T_1 or T_2 is singular.

3. Consider the following 2×2 elementary matrices:

   E_1 is obtained from the 2×2 identity matrix, I, by performing the elementary row operation $R_1 \leftrightarrow R_2$;

   E_2 is obtained from the 2×2 identity matrix, I, by performing the elementary row operation $aR_1 + R_2 \rightarrow R_2$, for some scalar a; and

   E_3 is obtained from the 2×2 identity matrix, I, by performing the elementary row operation $bR_2 \rightarrow R_2$, for a nonzero scalar b.

   Compute the determinants of the matrices E_1, E_2 and E_3.

4. Let E_1, E_2 and E_3 be the elementary matrices defined in Problem 3 and let B denote any 2×2 matrix. Verify that

$$\det(E_i B) = \det(E_i) \cdot \det(B) \quad \text{for } i = 1, 2, 3.$$

5. Let A denote any 2×2 matrix.

   Prove that A is invertible if and only if $\det(A) \neq 0$.

   If $\det(A) \neq 0$, give a formula for computing A^{-1} in terms of $\det(A)$ and the entries of A.