Assignment #8

Due on Friday, February 20, 2009

Read Section 1.8 on *Subspaces* in Messer (pp. 39–44).

Do the following problems

1. Given two subsets A and B of $\mathbb{R}^n$, the **union** of A and B, denoted by $A \cup B$, is the set which contains all vectors that are in either A or B; in symbols,

$$A \cup B = \{v \in \mathbb{R}^n \mid v \in A \text{ or } v \in B\}.$$

   (a) Prove that $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

   (b) Suppose that W_1 and W_2 are two subspaces of $\mathbb{R}^2$. Give an example that shows that $W_1 \cup W_2$ is not necessarily a subspace of $\mathbb{R}^2$.

2. Given two subsets A and B of $\mathbb{R}^n$, the **sum** of A and B, denoted by $A + B$, is the set which contains all vectors sums, $v + w$, such that $v \in A$ and $v \in B$; in symbols,

$$A + B = \{u \in \mathbb{R}^n \mid u = v + w, \text{ where } v \in A \text{ and } v \in B\}.$$

   Prove that if W_1 and W_2 are two subspaces of $\mathbb{R}^n$, then $W_1 + W_2$ is also a subspace of $\mathbb{R}^n$.

3. Let W_1 and W_2 be two subspaces of $\mathbb{R}^n$ and define $W_1 + W_2$ as in the previous problem. Prove that $W_1 \cap W_2$, W_1 and W_2 are subspaces of $W_1 + W_2$.

4. Let W_1 and W_2 be two subspaces of $\mathbb{R}^n$ and define $W_1 + W_2$ as in Problem 2 above. Suppose that $W_1 = \text{span}(S_1)$ and $W_2 = \text{span}(S_2)$, where $S_1 \subseteq W_1$ and $S_2 \subseteq W_2$. Prove that

$$W_1 + W_2 = \text{span}(S_1 \cup S_2).$$

5. Let S_1 and S_2 be two linearly independent subsets of $\mathbb{R}^n$. When can we say that $S_1 \cup S_2$ is linearly independent? Justify your answer.