Solutions to Review Problems for Exam 2

1. Let u denote a unit vector in $\mathbb{R}^n$ and define $f\colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$f(v) = \langle u, v\rangle u \quad \text{for all } v \in \mathbb{R}^n,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$.

(a) Verify that f is linear.

Solution: For $v, w \in \mathbb{R}^n$, compute

$$\begin{aligned} f(v+w) &= \langle u, v+w\rangle u \\ &= (\langle u, v\rangle + \langle u, w\rangle)u \\ &= \langle u, v\rangle u + \langle u, w\rangle u \\ &= f(v) + f(w). \end{aligned}$$

Similarly, for a scalar c and $v \in \mathbb{R}^n$,

$$\begin{aligned} f(cv) &= \langle u, cv\rangle u \\ &= c\langle u, v\rangle u \\ &= cf(v). \end{aligned}$$

□

(b) Give the image, $\mathcal{I}_f$, and null space, $\mathcal{N}_f$, of f, and compute $\dim(\mathcal{I}_f)$.

Solution: The image of f is the set

$$\mathcal{I}_f = \{w \in \mathbb{R}^n \mid w = f(v) \text{ for some } v \in \mathbb{R}^n\}.$$

We claim that $\mathcal{I}_f = \text{span}\{u\}$. To see why this is so, first observe that $f(u) = \langle u, u\rangle u = \|u\|^2 u = u$, since u is a unit vector. Thus, if $w \in \mathcal{I}_f$, then $w = cu$, for some scalar c, so that, by the linearity of f,

$$w = cf(u) = f(cu),$$

which shows that $w \in \mathcal{I}_f$. Thus

$$\text{span}\{u\} \subseteq \mathcal{I}_f. \tag{1}$$

Next, suppose that $w \in \mathcal{I}_f$; then, $w = f(v)$ for some $v \in \mathbb{R}^n$, so that

$$w = \langle u, v\rangle u \in \text{span}\{u\}.$$

Thus,

$$\mathcal{I}_f \subseteq \text{span}\{u\}. \tag{2}$$

Combining (1) and (2) yields that

$$\mathcal{I}_f = \text{span}\{u\}.$$

It then follows that

$$\dim(\mathcal{I}_f) = 1. \tag{3}$$

The null space of f is the set

$$\mathcal{N}_f = \{v \in \mathbb{R}^n \mid f(v) = \mathbf{0}\}.$$

Thus,

$$\begin{aligned} v \in \mathcal{N}_f \quad &\text{iff} \quad \langle u, v\rangle u = \mathbf{0} \\ &\text{iff} \quad \langle u, v\rangle = 0, \end{aligned}$$

since $u \neq \mathbf{0}$. It then follows that

$$\mathcal{N}_f = \{v \in \mathbb{R}^n \mid \langle u, v\rangle = 0\};$$

that is, $\mathcal{N}_f$ is the space of vectors which are orthogonal to u. □

(c) Use the Dimension Theorem for linear transformations $T\colon \mathbb{R}^n \to \mathbb{R}^m$,

$$\dim(\mathcal{N}_T) + \dim(\mathcal{I}_T) = n,$$

to compute $\dim(\mathcal{N}_f)$.

Solution: Using the dimension theorem and (3) we get that

$$\dim(\mathcal{N}_f) + 1 = n,$$

which implies that

$$\dim(\mathcal{N}_f) = n - 1.$$

□

2. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the linear transformation which maps the parallelogram spanned by

$$v_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

to the parallelogram spanned by

$$w_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(a) Give the matrix representation, M_T, relative to the standard basis in $\mathbb{R}^2$.

Solution: Assume that $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is linear and that $T(v_1) = w_1$ and $T(v_2) = w_2$. Writing v_1 and v_2 in terms of the standard basis in $\mathbb{R}^2$, we have that

$$v_1 = 2e_1 - e_2$$

and

$$v_2 = 2e_1 + e_2.$$

Thus, applying T and the linearity of T we then have that

$$2T(e_1) - T(e_2) = w_1 \tag{4}$$

and

$$2T(e_1) + T(e_2) = w_2. \tag{5}$$

We can solve (4) and (5) simultaneously to obtain that

$$T(e_1) = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

It then follows that the matrix representation, M_T, or T, relative to the standard basis in $\mathbb{R}^2$ is

$$M_T = [\ T(e_1) \quad T(e_2)\ ] = \begin{pmatrix} 0 & 1 \\ 1/2 & 0 \end{pmatrix}.$$

□

(b) Compute $\det(T)$. Does T preserve orientation?

Solution: Compute

$$\det(T) = \det(M_T) = -\frac{1}{2}.$$

Since, $\det(T) < 0$, T reverses orientation. □

(c) Show that T is invertible and compute the inverse of T.

Solution: Since $\det(T) \neq 0$, T is invertible, and the matrix representation for the inverse of T is given by

$$M_T^{-1} = \frac{1}{\det(T)} \begin{pmatrix} 0 & -1 \\ -1/2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.$$

Consequently, the inverse of T is given by

$$T^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ x \end{pmatrix}$$

for all $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. □

(d) Does T have real eigenvalues? If so, compute them and their corresponding eigenspaces.

Solution: The eigenvalues of T are scalars, λ, for which the system of equations

$$(M_T - \lambda I)v = \mathbf{0} \tag{6}$$

has nontrivial solutions. The system in (6) has nontrivial solutions if and only if the columns of the matrix

$$M_T - \lambda I = \begin{pmatrix} -\lambda & 1 \\ 1/2 & -\lambda \end{pmatrix}$$

are linearly dependent; this, in turn, is the case if and only if

$$\det(M_T - \lambda I) = 0,$$

or

$$\lambda^2 - \frac{1}{2} = 0.$$

Thus, $\lambda_1 = -\dfrac{1}{\sqrt{2}}$ and $\lambda_2 = \dfrac{1}{\sqrt{2}}$ are eigenvalues of T.

To find the eigespace corresponding to λ_1 we solve the homogenous system in (6) for $\lambda = \lambda_1$. We can do this by performing row operations of the augmented matrix

$$\left(\begin{array}{cc|c} \frac{1}{\sqrt{2}} & 1 & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \end{array}\right),$$

which is row–equivalent to the matrix

$$\left(\begin{array}{cc|c} 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Thus, the system in (6) for $\lambda = \lambda_1$ is equivalent to the homogeneous equation

$$x_1 + \sqrt{2}\, x_2 = 0,$$

which has solutions

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}.$$

Thus, the eigenspace of T associated with $\lambda_1 = -\dfrac{1}{\sqrt{2}}$ is

$$E_T(\lambda_1) = \text{span}\left\{ \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} \right\}.$$

Similarly, we can compute the eigenspace of T associated with $\lambda_2 = \dfrac{1}{\sqrt{2}}$ to be

$$E_T(\lambda_2) = \text{span}\left\{ \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \right\}.$$

□

3. Define $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(v) = Av \quad \text{for all } v \in \mathbb{R}^3,$$

where A is the 3×3 matrix given by

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}.$$

Find all eigenvalues and corresponding eigenspaces for the transformation T.

***Solution**:* First, observe that the third row of A is a multiple of the first and, therefore, A is singular. This implies that $\lambda = 0$ is an eigenvalue of A. to find the corresponding eigenspace, we solve the homogeneous system

$$Av = \mathbf{0} \tag{7}$$

for $v \in \mathbb{R}^3$. In order to do this, we reduce the augmented matrix

$$\left(\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 6 & -1 & 0 & 0 \\ -1 & -2 & -1 & 0 \end{array}\right)$$

to

$$\begin{pmatrix} 1 & 0 & 1/13 & | & 0 \\ 0 & 1 & 6/13 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

Thus the system in (7) is equivalent to

$$\begin{cases} x_1 & + \frac{1}{13}x_3 & = & 0 \\ & x_2 + \frac{6}{13}x_3 & = & 0, \end{cases}$$

which can be solved to yield the solutions

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}.$$

Thus, the eigenspace of A associated with $\lambda_1 = 0$ is

$$E_A(0) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix} \right\}.$$

Next, we see is A has other eigenvalues. In order to do this, we look for values of λ for which the homogeneous system

$$(A - \lambda I)v = \mathbf{0} \tag{8}$$

has nontrivial solutions. The system in (8) has nontrivial solutions if and only if $\det(A - \lambda I) = 0$, where

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix} \\ &= (1-\lambda)\begin{vmatrix} -1-\lambda & 0 \\ -2 & -1-\lambda \end{vmatrix} - 2\begin{vmatrix} 6 & 0 \\ -1 & -1-\lambda \end{vmatrix} + \begin{vmatrix} 6 & -1-\lambda \\ -1 & -2 \end{vmatrix} \\ &= (1-\lambda)(\lambda+1)^2 + 12(\lambda+1) - 12 - (\lambda+1) \\ &= -\lambda(\lambda+4)(\lambda-3). \end{aligned}$$

It then follows that $\lambda_1 = 0$, $\lambda_2 = -4$ and $\lambda_3 = 3$ are eigenvalues of A.

We have already compute $E_A(\lambda_1)$. To compute the eigenspace corresponding to λ_2, we solve the homogeneous system (8) with $\lambda = \lambda_2 = -4$. We do this by reducing the augmented matrix

$$\begin{pmatrix} 5 & 2 & 1 & | & 0 \\ 6 & 3 & 0 & | & 0 \\ -1 & -2 & 3 & | & 0 \end{pmatrix}$$

to

$$\begin{pmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & -2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Thus the system in (8) with $\lambda = -4$ is equivalent to

$$\begin{cases} x_1 \quad + \quad x_3 = 0 \\ \quad x_2 - 2x_3 = 0, \end{cases}$$

which can be solved to yield the solutions

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.$$

Thus, the eigenspace of A associated with $\lambda_2 = -4$ is

$$E_A(-4) = \text{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \right\}.$$

Similar calculations show that

$$E_A(3) = \text{span}\left\{ \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} \right\}.$$

□

4. Find a value of d for which the matrix

$$A = \begin{pmatrix} 1 & -2 \\ 3 & d \end{pmatrix}$$

is not invertible.

Show that, for that value of d, $\lambda = 0$ is an eigenvalue of A. Give the eigenspace corresponding to 0. What is the dimension of $E_A(0)$?

***Solution**:* The matrix A fails to be invertible when $\det(A) = 0$. This occurs when $d = -6$. For this value of d, the matrix A becomes

$$A = \begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix}$$

and observe that its second column is a multiple of the first. Therefore, the columns of A are linearly dependent; hence, the system

$$Av = \mathbf{0} \tag{9}$$

has nontrivial solutions and therefore $\lambda = 0$ is an eigenvalue of A. To find the corresponding eigenspace, observe that the system in (9) is equivalent to the equation

$$x_1 - 2x_2 = 0,$$

which has solutions

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Thus, the eigenspace of A associated with $\lambda = 0$ is

$$E_A(0) = \text{span}\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}.$$

Therefore, $\dim(E_A(0)) = 1$. □

5. Use the fact that $\det(AB) = \det(A)\det(B)$ for all $A, B \in \mathbb{M}(n, n)$ to compute $\det(A^{-1})$, provided that A is invertible.

   Proof: Assume that A is invertible with inverse A^{-1}. Then,

   $$A^{-1}A = I,$$

   where I is the $n \times n$ identity matrix. Taking determinants on both sides of the equation yields that

   $$\det(A^{-1}A) = 1,$$

   from which we get that

   $$\det(A^{-1})\det(A) = 1.$$

   This, since $\det(A) \neq 0$ because A is invertible, we get that

   $$\det(A^{-1}) = \frac{1}{\det(A)}.$$

   □

6. Let A and B be $n \times n$ matrices. Show that if AB is invertible, then so is A.

   Proof: Suppose that AB is invertible. Then, there exists an $n \times n$ matrix, C, such that
   $$(AB)C = I,$$
   where I is the $n \times n$ identity matrix. Thus, by associativity of matrix multiplication,
   $$A(BC) = I,$$
   which shows that A has a right–inverse and is therefore invertible. □

7. Let A be a 3×3 matrix satisfying $A^3 - 6A^2 - 2A + 12I = O$, where I is the 3×3 identity matrix and O is the 3×3 zero matrix.

   (a) Prove that A is invertible and given a formula for computing its inverse in terms of I, A and A^2.

   Solution: We can solve the equation $A^3 - 6A^2 - 2A + 12I = O$ for $12I$ and then divide by 12 to get that
   $$A\left(\frac{1}{6}I + \frac{1}{2}A - \frac{1}{12}A^2\right) = I,$$
   which shows that A has a right–inverse and is therefore invertible with
   $$A^{-1} = \frac{1}{6}I + \frac{1}{2}A - \frac{1}{12}A^2.$$
   □

   (b) Prove that if λ is an eigenvalue of A, then $\lambda^3 - 6\lambda^2 - 2\lambda + 12 = 0$. Deduce therefore that λ is one of 6, $\sqrt{2}$ or $-\sqrt{2}$.

   Proof: Let λ be an eigenvalue of A. Then, there exists a nonzero vector, v, in $\mathbb{R}^3$ such that
   $$Av = \lambda v.$$
   Multiplying on both sides by A we then get that
   $$A^2 v = \lambda A v = \lambda(\lambda v) = \lambda^2 v.$$
   Multiplying the last equation by A we then get that
   $$A^3 v = \lambda^3 v.$$

Thus, applying $A^3 - 6A^2 - 2A + 12I = O$ to to v we get that

$$(A^3 - 6A^2 - 2A + 12I)v = Ov,$$

which, by the distributive property, implies that

$$A^3v - 6A^2v - 2Av + 12v = \mathbf{0}.$$

Thus,

$$\lambda^3v - 6\lambda^2v - 2\lambda v + 12v = \mathbf{0},$$

or

$$(\lambda^3 - 6\lambda^2 - 2\lambda + 12)v = \mathbf{0},$$

from which we get that

$$\lambda^3 - 6\lambda^2 - 2\lambda + 12 = 0,$$

since v is nonzero.

Observe that $\lambda^3 - 6\lambda^2 - 2\lambda + 12$ factors into $(\lambda - 6)(\lambda + \sqrt{2})(\lambda - \sqrt{2})$. □

8. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(v) = Av$ for all $v \in \mathbb{R}^2$, where A is a 2×2 matrix. Let $\text{area}(P(v_1, v_2))$ denote the area of the parallelogram determined by the vectors v_1 and v_2. Prove that

$$\text{area}(P(T(v_1), T(v_2))) = |\det(A)| \cdot \text{area}(P(v_1, v_2)).$$

Solution: Observe that the matrix $[\ T(v_1) \quad T(v_2)\ ] = [\ Av_1 \quad Av_2\ ]$ can be written as

$$[\ T(v_1) \quad T(v_2)\ ] = A[\ v_1 \quad v_2\ ],$$

by the definition of the matrix product. Thus, taking the determinant on both sides we have

$$\begin{aligned} \det([\ T(v_1) \quad T(v_2)\ ]) &= \det(A[\ v_1 \quad v_2\ ]) \\ &= \det(A)\det([\ v_1 \quad v_2\ ]). \end{aligned}$$

Thus, taking the absolute value on both sides,

$$\text{area}(P(T(v_1), T(v_2))) = |\det(A)| \cdot \text{area}(P(v_1, v_2)).$$

□