## Review Problems for Final Exam

1. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Prove that $T$ is singular if and only if $\lambda = 0$ is an eigenvalue of $T$.

2. Let $B$ be an $n\times n$ matrix satisfying $B^3 = 0$ and put $A = I + B$, where $I$ denotes the $n\times n$ identity matrix. Prove that $A$ is invertible and compute $A^{-1}$ in terms of $I$, $B$ and $B^2$.

3. Let $A = \begin{pmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{pmatrix}$.

   (a) Find a basis for $\mathbb{R}^2$ made up of eigenvectors of $A$.

   (b) Let $Q$ be the $2\times 2$ matrix $Q = [\, v_1 \quad v_2 \,]$, where $\{v_1, v_2\}$ is the basis of eigenvectors found in (a) above. Verify that $Q$ is invertible and compute $Q^{-1}AQ$. What do you discover?

   (c) Use the result in part (b) above to find a formula for for computing $A^k$ for every positive integer $k$. Can you say anything about $\lim\limits_{k\to\infty} A^k$?

4. Let $A$ be an $m\times n$ matrix and $b\in\mathbb{R}^m$. Prove that if $Ax = b$ has a solution $x$ in $\mathbb{R}^n$, then $\langle b, v\rangle = 0$ for every $v$ is the null space of $A^T$.

5. Let $A$ be an $m\times n$ matrix. Prove that if $A^T$ is nonsingular, then $Ax = b$ has a solution $x$ in $\mathbb{R}^n$ for every $b\in\mathbb{R}^n$.

6. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ denote a linear transformation. Prove that if $\lambda$ is an eigenvalue of $T$, then $\lambda^k$ is an eigenvalue of $T^k$ for every positive integer $k$. If $\mu$ is an eigenvalue of $T^k$, is $\mu^{1/k}$ always and eigenvalue of $T$?

7. Let $\mathcal{E} = \{e_1, e_2\}$ denote the standard basis in $\mathbb{R}^2$, and let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear function satisfying: $f(e_1) = e_1 + e_2$ and $f(e_2) = 2e_1 - e_2$.

   Give the matrix representation for $f$ and $f\circ f$ relative to $\mathcal{E}$.

8. A function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined as follows: Each vector $v\in\mathbb{R}^2$ is reflected across the $y$–axis, and then doubled in length to yield $f(v)$.

   Verify that $f$ is linear and determine the matrix representation, $M_f$, for $f$ relative to the standard basis in $\mathbb{R}^2$.

9. Find a $2\times 2$ matrix $A$ such that the function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(v) = Av$ maps the coordinates of any vector, relative to the standard basis in $\mathbb{R}^2$, to its coordinates relative the basis $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.