Solutions Review Problems for Final Exam

1. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Prove that T is singular if and only if $\lambda = 0$ is an eigenvalue of T.

   ***Solution**:* T is singular if and only if

   $$T(v) = \mathbf{0}$$

   has nontrivial solutions; thus, T is singular if and only if

   $$T(v) = 0v$$

   has nontrivial solutions. Consequently, T is singular if and only if $\lambda = 0$ is an eigenvalue of T. □

2. Let B be an $n \times n$ matrix satisfying $B^3 = O$ and put $A = I + B$, where I denotes the $n \times n$ identity matrix. Prove that A is invertible and compute A^{-1} in terms of I, B and B^2.

   ***Solution**:* Consider the matrix $Q = c_1 I + c_2 B + c_3 B^2$ and look for scalars c_1, c_2 and c_3 such that $AQ = I$.
   Now,

   $$\begin{aligned} AQ &= (I+B)Q \\ &= c_1 I + c_2 B + c_3 B^2 + B(c_1 I + c_2 B + c_3 B^2) \\ &= c_1 I + c_2 B + c_3 B^2 + c_1 B + c_2 B^2 + c_3 B^3 \\ &= c_1 I + (c_1 + c_2)B + (c_2 + c_3)B^2, \end{aligned}$$

   since $B^3 = O$. Thus, $AQ = I$ if and only if

   $$\begin{cases} c_1 &= 1 \\ c_1 + c_2 &= 0 \\ c_2 + c_3 &= 0. \end{cases}$$

   Solving this system we get $c_1 = 1$, $c_2 = -1$ and $c_3 = 1$. Hence, if $Q = I - B + B^2$, then Q is a right–inverse of $A = I + B$ and therefore $A = I + B$ is invertible and $A^{-1} = I - B + B^2$. □

3. Let $A = \begin{pmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{pmatrix}$.

   (a) Find a basis for $\mathbb{R}^2$ made up of eigenvectors of A.

   Solution: First, we look for values of λ such that the system

$$(A - \lambda I)v = \mathbf{0} \tag{1}$$

has nontrivial solutions in $\mathbb{R}^2$. This is the case if and only if $\det(A - \lambda I) = 0$, which occurs if and only if

$$\lambda^2 - \frac{7}{6}\lambda + \frac{1}{6} = 0,$$

or

$$(\lambda - 1)\left(\lambda - \frac{1}{6}\right) = 0.$$

We then get that

$$\lambda_1 = \frac{1}{6} \quad \text{and} \quad \lambda_2 = 1$$

are eigenvalues of A.

To find an eigenvector corresponding to the eigenvalue λ_1, we solve the system in (1) for $\lambda = \lambda_1$. In this case, the system can be reduced to the equation

$$x_1 + x_2 = 0,$$

which has solutions

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

where t is arbitrary. We can therefore take

$$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

as an eigenvector corresponding to $\lambda = \frac{1}{6}$.

Similar calculations for $\lambda = \lambda_2 = 1$ lead to the equation

$$3x_1 - 2x_2 = 0,$$

which has solutions

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

where t is arbitrary. Thus, in this case, we obtain the eigenvector

$$v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Since v_1 and v_2 are linearly independent, the constitute a basis for $\mathbb{R}^2$ because $\dim(\mathbb{R}^2) = 2$. □

(b) Let Q be the 2×2 matrix $Q = [\, v_1 \quad v_2 \,]$, where $\{v_1, v_2\}$ is the basis of eigenvectors found in (a) above. Verify that Q is invertible and compute $Q^{-1}AQ$. What do you discover?

***Solution**:* $Q = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$, so that $\det(Q) = 3 + 2 = 5 \neq 0$. Hence Q is invertible and

$$Q^{-1} = \frac{1}{5} \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}.$$

Next, compute

$$\begin{aligned} Q^{-1}AQ &= \frac{1}{5} \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1/6 & 2 \\ -1/6 & 3 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 5/6 & 0 \\ 0 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 1/6 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \end{aligned}$$

Thus, $Q^{-1}AQ$ is a diagonal matrix with the eigenvalues as entries along the main diagonal. □

(c) Use the result in part (b) above to find a formula for for computing A^k for every positive integer k. Can you say anything about $\lim_{k\to\infty} A^k$?

***Solution**:* Let D denote the matrix $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Then, from part (b) in this problem,

$$Q^{-1}AQ = D.$$

Multiplying this equation by Q on the left and Q^{-1} on the right, we obtain that

$$A = QDQ^{-1}.$$

It then follows that

$$\begin{aligned} A^2 &= (QDQ^{-1})(QDQ^{-1}) \\ &= QD(Q^{-1}Q)DQ^{-1} \\ &= QDIDQ^{-1} \\ &= QD^2Q^{-1}. \end{aligned}$$

We may now proceed by induction on k to show that

$$A^k = QD^kQ^{-1} \quad \text{for all} \quad k = 1, 2, 3, \ldots$$

In fact, once we have established that

$$A^{k-1} = QD^{k-1}Q^{-1},$$

we compute

$$\begin{aligned} A^k &= AA^{k-1} \\ &= (QDQ^{-1})(QD^{k-1}Q^{-1}) \\ &= QD(Q^{-1}Q)D^{k-1}Q^{-1} \\ &= QDID^{k-1}Q^{-1} \\ &= QD^kQ^{-1}. \end{aligned}$$

Thus, we may compute A^k as follows

$$
\begin{aligned}
A^k &= QD^kQ^{-1} \\
&= \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^k \frac{1}{5} \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} \\
&= \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} \\
&= \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1/6^k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} . \\
&= \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 3/6^k & -2/6^k \\ 1 & 1 \end{pmatrix} \\
&= \frac{1}{5} \begin{pmatrix} (3/6^k) + 2 & -(2/6^k) + 2 \\ -(3/6^k) + 3 & (2/6^k) + 3 \end{pmatrix}
\end{aligned}
$$

Observe that, as $k \to \infty$,

$$
A^k \to \begin{pmatrix} 2/5 & 2/5 \\ 3/5 & 3/5 \end{pmatrix}.
$$

□

4. Let A be an $m \times n$ matrix and $b \in \mathbb{R}^m$. Prove that if $Ax = b$ has a solution x in $\mathbb{R}^n$, then $\langle b, v \rangle = 0$ for every v is the null space of A^T.

***Solution**:* Let x be a solution of $Ax = b$ and $v \in \mathcal{N}_{A^T}$. Then, $A^T v = \mathbf{0}$ and

$$
\begin{aligned}
\langle b, v \rangle &= \langle Ax, v \rangle \\
&= (Ax)^T v \\
&= x^T A^T v \\
&= x^T \mathbf{0} \\
&= 0.
\end{aligned}
$$

□

5. Let A be an $m \times n$ matrix. Prove that if A^T is nonsingular, then $Ax = b$ has a solution x in $\mathbb{R}^n$ for every $b \in \mathbb{R}^n$.

   Solution: If A^T is nonsingular, then the null–space, $\mathcal{N}_{A^T}$, is the trivial subspace, $\{\mathbf{0}\}$, of $\mathbb{R}^m$. Consequently, $\dim(\mathcal{N}_{A^T}) = 0$. Thus, by the Dimension Theorem for Matrices, the rank of A^T is m, since $A^T \in \mathbb{M}(n, m)$. Thus, since the rank of A^T is the same as the rank of A, by the equality of the column and row ranks, it follows that $A \in \mathbb{M}(m, n)$ has rank m. In other words, the dimension of the column space of A is m. Thus, since the column space of A, $\mathcal{C}_A$, is a subspace of $\mathbb{R}^m$, it follows that

$$\mathcal{C}_A = \mathbb{R}^m.$$

   Therefore, if $A = [\, v_1 \quad v_2 \quad \cdots \quad v_n \,]$, where $v_1, v_2, \ldots, v_n$ are the columns of A , then for any $b \in \mathbb{R}^m$, there exist scalars $c_1, c_2, \ldots, c_n$ such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = b,$$

   or

$$[\, v_1 \quad v_2 \quad \cdots \quad v_n \,] \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = b,$$

   which implies that the system

$$Ax = b$$

   has a solution. □

6. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ denote a linear transformation. Prove that if λ is an eigenvalue of T, then λ^k is an eigenvalue of T^k for every positive integer k. If μ is an eigenvalue of T^k, is $\mu^{1/k}$ always and eigenvalue of T?

   Solution: Let λ be an eigenvalue of $T\colon \mathbb{R}^n \to \mathbb{R}^n$. Then, there exists a nonzero vector, v, in $\mathbb{R}^n$ such that

$$T(v) = \lambda v.$$

   Applying the transformation, T, on both sides and using the fact that T is linear and that v is an eigenvector corresponding to λ, we obtain that

$$T^2(v) = T(\lambda v) = \lambda T(v) = \lambda \lambda v = \lambda^2 v,$$

so that, since $v \neq \mathbf{0}$, λ^2 is an eigenvalue for T^2.

We may now proceed by induction on k to show that

$$\lambda^k, \quad \text{for all } \; k = 1, 2, 3, \ldots,$$

is an eigenvalue of T^k. To do this, assume we have established that λ^{k-1} is an eigenvalue of T^{k-1} and that v is an eigenvector for T corresponding to the eigenvalue λ, so that v is also an eigenvector of T^{k-1} corresponding to λ^{k-1}. We then have that

$$T^{k-1}(v) = \lambda^{k-1} v.$$

Thus, applying the transformation, T, on both sides and using the fact that T is linear and that v is an eigenvector corresponding to λ, we obtain that

$$T^k(v) = T(T^{k-1}v) = T(\lambda^{k-1}v) = \lambda^{k-1}T(v) = \lambda^{k-1}\lambda v = \lambda^k v,$$

so that, since $v \neq \mathbf{0}$, λ^k is an eigenvalue for T^k.

Next, consider the function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by rotation in the counterclockwise sense by 90° or $\pi/2$ radians; that is,

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

Then, $T^2\colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$T^2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,$$

which has $\mu = -1$ as the only eigenvalue. Observe that T has no real eigenvalues, so $\mu^{1/2}$ cannot be a (real) eigenvalue of T. □

7. Let $\mathcal{E} = \{e_1, e_2\}$ denote the standard basis in $\mathbb{R}^2$, and let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear function satisfying: $f(e_1) = e_1 + e_2$ and $f(e_2) = 2e_1 - e_2$.

   Give the matrix representation for f and $f \circ f$ relative to $\mathcal{E}$.

   ***Solution**:* Observe that

$$f(e_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad f(e_2) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

It then follows that the matrix representation for f relative to $\mathcal{E}$ is

$$M_f = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}.$$

The matrix representation of $f \circ f$ is the product $M_f M_f$, or

$$M_{f\circ f} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}.$$

□

8. A function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined as follows: Each vector $v \in \mathbb{R}^2$ is reflected across the y–axis, and then doubled in length to yield $f(v)$.

   Verify that f is linear and determine the matrix representation, M_f, for f relative to the standard basis in $\mathbb{R}^2$.

   ***Solution**:* The function f is the composition of the reflection $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$R\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,$$

   and the function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(w) = 2w$ for all $w \in \mathbb{R}^2$ or, in matrix form,

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

   Note that both R and T are linear since they are both defined in terms of multiplication by matrix. It then follows that $f = T \circ R$ is linear and its matrix representation, M_f, relative to the standard basis in $\mathbb{R}^2$ is

$$M_f = M_T M_R = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

□

9. Find a 2×2 matrix A such that the function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(v) = Av$ maps the coordinates of any vector, relative to the standard basis in $\mathbb{R}^2$, to its coordinates relative the basis $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.

***Solution**:* Denote the vectors in $\mathcal{B}$ by v_1 and v_2, respectively, so that

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We want the function T to satisfy

$$T(v) = [v]_{\mathcal{B}}$$

for every $v \in \mathbb{R}^2$ given in terms of the standard basis in $\mathbb{R}^2$. We want T to be linear, so that all we need to know about T is what it does to the standard basis; that is, we need to know $T(e_1)$ and $T(e_2)$. To find out what $T(e_1)$ is, we need to find scalars c_1 and c_2 such that

$$c_1 v_1 + c_2 v_2 = e_1$$

. That is, we need to solve the system

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e_1,$$

which we can solve by multiplying by the inverse of the matrix on the left:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} e_1 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix},$$

so that

$$T(e_1) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

Similarly,

$$T(e_2) = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}.$$

It then follows that

$$A = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}.$$

□