Assignment #14

Due on Wednesday, April 14, 2010

Do the following problems.

1. Let $x_n = \dfrac{1}{\sqrt{n-1}}$ for $n \geqslant 2$. Show that (x_n) converges and compute its limit.

2. Let (x_n) be a sequence of real numbers satisfying $x_n \geqslant 0$ for all $n \in \mathbb{N}$ and define $y_n = \sqrt{x_n}$ for all $n \in \mathbb{N}$. Suppose that (x_n) converges to 0. Prove that the sequence (y_n) converges and compute its limit.

3. Let $a > 0$ and define $x_n = a^{1/n}$ for all $n \in \mathbb{N}$. In this problem and the next you will prove that $\lim\limits_{n\to\infty} x_n = 1$.

   First, assume that $a > 1$.

   (a) Prove that $x_n > 1$ for all $n \in \mathbb{N}$.

   (b) Put $y_n = x_n - 1$ for all $n \in \mathbb{N}$. Then $a = (1 + y_n)^n$ for all $n \in \mathbb{N}$. Apply the Binomial Theorem to show that
   $$a > ny_n \quad \text{for all } n \in \mathbb{N}. \tag{1}$$

   (c) Use the inequality in (1) to prove that (y_n) converges to 0. Deduce that (x_n) converges to 1.

   (d) Prove that if $a > 1$, then $\lim\limits_{n\to\infty} a^{1/n} = 1$.

4. Prove that for any positive real number, a, $\lim\limits_{n\to\infty} a^{1/n} = 1$.

5. In this problem you will prove that $\lim\limits_{n\to\infty} n^{1/n} = 1$.

   (a) Set $y_n = n^{1/n} - 1$ and note that $n = (1 + y_n)^n$. Use the Binomial Theorem to show that
   $$n > \frac{n(n-1)}{2}\, y_n^2 \quad \text{for all } n \geqslant 2. \tag{2}$$

   (b) Use the inequality in (2) to prove that (y_n) converges and compute its limit.

   (c) Deduce that $\lim\limits_{n\to\infty} n^{1/n} = 1$.