Assignment #2

Due on Monday, February 1, 2010

Read Section 4.3 on *Well–Ordering and Induction* on pp. 54–57 in Schramm's text.

Read Section 4.5 on *Strong Induction* on pp. 58–60 in Schramm's text.

Do the following problems

1. Let P, Q and R denote propositions. Use a truth–table to verify that the implication $P \Rightarrow (Q \vee R)$ is logically equivalent to $(P \wedge \neg Q) \Rightarrow R$.

2. Let m and n denote integers. Prove that if 2 divides mn, then either 2 divides m or 2 divides n.

   Suggestion: Use the result of the previous problem and prove the implication: If 2 divides mn and 2 does not divide m, then 2 divides n.

3. Use mathematical induction to prove that every non–empty subset of the natural numbers must have a smallest element.

   Suggestion: Let A denote a non–empty subset of $\mathbb{N}$. We claim that A must have a smallest element. Argue by contradiction: Assume that A has no smallest element and let S denote the set of natural numbers that are not in A.

   (a) Prove that $1 \in S$.

   (b) Prove that $k \in S$ for all $k \in \{1, 2, \ldots, n\}$ implies that $n + 1 \in S$.

   (c) Deduce that $S = \mathbb{N}$. Explain why this is a contradiction.

4. Find the smallest natural number that can be written as the sum of three prime numbers, but cannot be written as the sum of two prime numbers.

5. Let $m, n \in \mathbb{Z}$ with $0 < m < n$. Define $S = \{n - km \mid k \in \mathbb{Z} \text{ and } n - mk \geqslant 0\}$.

   (a) Prove that S has a smallest element and call it r.

   (b) Prove that $r \in \{0, 1, \ldots m - 1\}$.
   Suggestion: Show that $r \geqslant m$ is impossible.

   (c) Prove: Given positive integers, m and n, with $m < n$, there exist unique integers, q and r, such that,

   $$n = qm + r \quad \text{where} \quad r \in \{0, 1, \ldots m - 1\}.$$

   Note: This is a special case of the Division Algorithm.