Solutions to Exam 1 (Part II)

1. For a subset, A, of the real numbers and a real number, c, define the following sets

   (i) $A + c = \{y \in \mathbb{R} \mid y = x + c \text{ where } x \in A\}$, and

   (ii) $cA = \{y \in \mathbb{R} \mid y = cx \text{ where } x \in A\}$.

   Prove the following statements.

   (a) If A is non–empty and bounded above, then

   $$\sup(A + c) = \sup A + c.$$

   Proof: First, note that, since $A \neq \emptyset$, there exists $a \in A$. It then follows that $a + c \in A + c$, which shows that $A + c$ is non–empty.

   Next, observe that from the fact that A is bounded above we obtain that $\sup(A)$ exists, by the completeness axiom. Furthermore,

   $$a + c \leqslant \sup(A) + c \quad \text{for all } \ a \in A,$$

   which shows that $A + c$ is bounded above by $\sup(A) + c$. Therefore, by the completeness axiom, again, $\sup(A + c)$ exists and

   $$\sup(A + c) \leqslant \sup(A) + c. \tag{1}$$

   Also,

   $$a + c \leqslant \sup(A + c) \quad \text{for all } \ a \in A,$$

   which implies that

   $$a \leqslant \sup(A + c) - c \quad \text{for all } \ a \in A;$$

   that is, $\sup(A + c) - c$ is an upper bound for A. Consequently,

   $$\sup(A) \leqslant \sup(A + c) - c,$$

   which implies that

   $$\sup(A) + c \leqslant \sup(A + c). \tag{2}$$

   Combining the inequalities in (1) and (2) yields the result. □

   (b) If A is non–empty and bounded above and $c > 0$, then

   $$\sup(cA) = c \sup A.$$

Proof: First, note that, since $A \neq \emptyset$, there exists $a \in A$. It then follows that $ca \in cA$, which shows that cA is non–empty.

Next, observe that from the fact that A is bounded above we obtain that $\sup(A)$ exists, by the completeness axiom. Furthermore,

$$ca \leqslant c\sup(A) \quad \text{for all } \ a \in A,$$

since $c > 0$. Thus, cA is bounded above by $c\sup(A)$. Therefore, by the completeness axiom, $\sup(cA)$ exists and

$$\sup(cA) \leqslant c\sup(A). \tag{3}$$

Also,

$$ca \leqslant \sup(cA) \quad \text{for all } \ a \in A.$$

Multiplying both sides of the inequality by $c^{-1} > 0$ yields

$$a \leqslant c^{-1}\sup(cA) \quad \text{for all } \ a \in A;$$

which shows that $c^{-1}\sup(cA)$ is an upper bound for A. Consequently,

$$\sup(A) \leqslant c^{-1}\sup(cA),$$

which implies that

$$c\sup(A) \leqslant \sup(cA). \tag{4}$$

Combining the inequalities in (3) and (4) yields the result. □

(c) What happens if $c < 0$ in part (b)? State and prove your result.

Solution:

Claim: Suppose that A is non–empty and bounded above. If $c < 0$, then cA is nonempty and bounded below, and

$$\inf(cA) = c\sup(A).$$

Proof of Claim: First note that, since $c < 0$, from

$$a \leqslant \sup(A) \quad \text{for all } \ a \in A,$$

we get that

$$ca \geqslant c\sup(A) \quad \text{for all } \ a \in A,$$

which shows that cA is bounded below by $c\sup(A)$. Since cA is non–empty, as seen in part (b), $\inf(cA)$ exists and

$$c\sup(A) \leqslant \inf(cA). \tag{5}$$

Next, from

$$\inf(cA) \leqslant ca \quad \text{for all } \; a \in A,$$

we get that

$$c^{-1}\inf(cA) \geqslant a \quad \text{for all } \; a \in A,$$

since $c^{-1} < 0$. Consequently, $c^{-1}\inf(cA)$ is an upper bound for A, and therefore

$$\sup(A) \leqslant c^{-1}\inf(cA).$$

Multiplying by $c < 0$ on both sides then yields

$$c\sup(A) \geqslant \inf(cA).$$

Combining this inequality with the one in (5) yields the claim. □

□

***Alternate Solution**:* Using the result from part (b) we have that

$$\sup(|c|A) = |c|\sup(A),$$

since $|c| > 0$. Thus, multiplying by -1 on both sides of the last equation,

$$-\sup(|c|A) = c\sup(A),$$

where we have used the fact that $-|c| = c$ if $c < 0$. On the other hand,

$$-\sup(|c|A) = \inf(-|c|A) = \inf(cA).$$

Thus, if $c < 0$ and $A \neq \emptyset$ is bounded above, cA is bounded below and

$$\inf(cA) = c\sup(A).$$

□

2. Let $A = \left\{ \dfrac{n+1}{n} \;\middle|\; n \in \mathbb{N} \right\}$.

   Prove that A is bounded above and below and compute $\inf A$ and $\sup A$.

   Justify your calculations and prove any assertion you make.

Solution: Given any $a \in A$, there exists $n \in \mathbb{N}$ such that

$$a = \frac{n+1}{n} = 1 + \frac{1}{n}.$$

It then follows that

$$1 < a \leqslant 2 \quad \text{for all } a \in A.$$

Thus, 1 is a lower bound for A and 2 is an upper bound. We then have that $\inf(A)$ and $\sup(A)$ exist, with

$$1 \leqslant \inf(A) \quad \text{and} \quad \sup(A) \leqslant 2.$$

Since $2 = 1 + \dfrac{1}{1}$ is in A, it follows that

$$\sup(A) = 2.$$

To show that $\inf(A) = 1$, assume by way of contradiction that

$$\inf(A) > 1.$$

Then, $\inf(A) - 1 > 0$ and therefore $\dfrac{1}{\inf(A) - 1} > 0$. Next, use the fact that $\mathbb{N}$ is not bounded above to obtain $m \in \mathbb{N}$ such that

$$\frac{1}{\inf(A) - 1} < m,$$

from which we get that

$$\inf(A) - 1 > \frac{1}{m},$$

or

$$\inf(A) > 1 + \frac{1}{m},$$

where $1 + \dfrac{1}{m} \in A$. This is a contradiction; therefore,

$$\inf(A) = 1.$$

$\square$

Alternate Solution: Let $B = \left\{ \dfrac{1}{n} \;\middle|\; n \in \mathbb{N} \right\}$. We have seen in class that $\sup(B) = 1$ and $\inf(B) = 0$.
Observe that

$$A = B + 1.$$

Thus, by the result of Problem 1(a) in this exam,

$$\sup(A) = \sup(B) + 1 = 1 + 1 = 2.$$

We can also prove that

$$\inf(A) = \inf(B) + 1 = 0 + 1 = 1.$$

□