Solutions to Exam 1 (Part I)

1. Provide concise answers to the following questions:

   (a) A subset, A, of the real numbers is said to be **bounded** if there exists a positive real number, M, such that

   $$|a| \leqslant M \quad \text{for all } \ a \in A.$$

   Give the negation of the statement

   "A is bounded."

   Answer*:* The negation of "A *is bounded*" is

   For every positive number, M, there exists and an element, a, in A such that $|a| > M$.

   □

   (b) Let A denote a subset of the real numbers and β a positive real number. Give the contrapositive for the following implication:

   $$t \in A \Rightarrow t \leq s - \beta.$$

   Answer*:* The contrapositive of "$t \in A \Rightarrow t \leq s - \beta$" is

   $$t > s - \beta \Rightarrow t \notin A.$$

   □

2. Use the field and order axioms of the real numbers to prove the following.

   (a) Let $a, b \in R$. If $ab = 0$, then either $a = 0$ or $b = 0$.

   Proof: Assume that $ab = 0$ and $a \neq 0$. Then, by Field Axiom (F_9), a^{-1} exists. Multiplying

   $$ab = 0$$

   by a^{-1} on both sides yields

   $$a^{-1}(ab) = a^{-1} \cdot 0 = 0,$$

   from which we get that $b = 0$, where we have used the Field Axioms (F_7), (F_9) and (F_{10}). □

(b) Let $p \in \mathbb{R}$. If $p > 1$, then $p < p^2$.

Proof: Assume that $p > 1$. It then follows that $p > 0$, since $1 > 0$. We also have that $p - 1 > 0$. Consequently, by the Order Axiom (O_3),

$$p(p-1) > 0.$$

Thus, by the distributive property,

$$p^2 - p > 0,$$

from which we get that $p < p^2$. □

3. Use the completeness axiom of $\mathbb{R}$ to prove that the set of natural numbers is not bounded above. Deduce, therefore, that for any real number, x, there exists a natural number, n, such that

$$x < n.$$

Proof: Assume by way of contradiction that $\mathbb{N}$ is a bounded above. Then, since $\mathbb{N}$ is not empty, it follows from the completeness axiom that $\sup(\mathbb{N})$ exists. Thus there must be $m \in \mathbb{N}$ such that

$$\sup(\mathbb{N}) - 1 < m. \tag{1}$$

It follows from the inequality in (1) that

$$\sup(\mathbb{N}) < m + 1,$$

where $m + 1 \in \mathbb{N}$. This is a contradiction. Therefore, it must be that case that $\mathbb{N}$ not bounded above.

Thus, given any real number, x, there must be a natural number, n, such that

$$x < n.$$

Otherwise,

$$m \leqslant x \quad \text{for all } m \in \mathbb{N},$$

which would say that x is an upper bound for $\mathbb{N}$. But we just proved that $\mathbb{N}$ is not bounded above. □