Solutions to Exam 2 (Part I)

1. Let (x_n) denote a sequence of real numbers.

   (a) State precisely what the statement "(x_n) converges" means.

   Answer: "(x_n) converges" means that there exists some $x \in \mathbb{R}$ such that, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

   $$n \geqslant N \Rightarrow |x_n - x| < \varepsilon.$$

   $\square$

   (b) Let $x_n = \dfrac{1}{\sqrt{n}}$ for all $n \in \mathbb{N}$. Use the definition that you stated in the previous part to prove that (x_n) converges.

   Solution: We show that (x_n) converges to 0.
   Let $\varepsilon > 0$ be given; then, $\varepsilon^2 > 0$. Thus, by the Archimedean Property, there exists $N \in \mathbb{N}$ such that

   $$\frac{1}{N} < \varepsilon^2.$$

   Thus, $n \geqslant N$ implies that

   $$0 < \frac{1}{\sqrt{n}} \leqslant \frac{1}{\sqrt{N}} < \varepsilon.$$

   Hence,

   $$n \geqslant N \Rightarrow \left| \frac{1}{\sqrt{n}} - 0 \right| < \varepsilon.$$

   This completes the proof of the fact that (x_n) converges to 0. $\square$

2. Let (x_n) denote a sequence of real numbers.

   (a) State precisely what it means for (x_n) to be a Cauchy sequence.

   Answer: (x_n) is a Cauchy sequence means that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

   $$m, n \geqslant N \Rightarrow |x_n - x_m| < \varepsilon.$$

   $\square$

(b) Prove that if (x_n) converges, the it is a Cauchy sequence.

Proof. Assume that (x_n) converges to $x \in \mathbb{R}$. Then, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n \geqslant N \Rightarrow |x_n - x| < \frac{\varepsilon}{2}.$$

Thus, using the triangle inequality, we get that $n, m \geqslant N$ implies that

$$\begin{aligned} |x_n - x_m| &= |x_n - x + x - x_m| \\ &\leqslant |x_n - x| + |x_m - x| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

We have therefore shown that (x_n) is a Cauchy sequence. $\square$

3. Let $B \subseteq \mathbb{R}$ be a non–empty subset which is bounded below and put $\ell = \inf B$.

(a) Prove that there exists a sequence of numbers in B which converges to ℓ.

Proof: Assume that B is a subset of $\mathbb{R}$ which is non–empty and bounded below. Thus, $\inf B$ exists. Set $\ell = \inf B$. Then, for every $n \in \mathbb{N}$, there exists $x_n \in B$ such that

$$x_n < \ell + \frac{1}{n};$$

otherwise $\ell + \dfrac{1}{n}$ would be a lower bound of B which is bigger than ℓ, which is impossible. We then have that

$$\ell \leqslant x_n < \ell + \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

It then follows by the Squeeze Theorem for sequences that (x_n) converges to ℓ. $\square$

(b) Apply the result of the previous part to the set

$$B = \{q \in \mathbb{Q} \mid q > 0 \text{ and } q^2 > 2\}$$

to deduce that there exists a sequence of rational numbers $\{q_n\}$ which converges to $\sqrt{2}$.

Note: You will need to prove that $\inf B = \sqrt{2}$.

Solution: First note that B is non–empty and bounded below. In fact, $2 \in B$ since $2 > 0$ and $2^2 = 4 > 2$. Furthermore, for any $q \in B$ we have that

$$q > 0 \text{ and } q^2 > 2,$$

which implies that $q > \sqrt{2}$. It then follows that $\sqrt{2}$ is a lower bound for B. Thus, $\inf B$ exists and

$$\sqrt{2} \leqslant \inf B.$$

To see that $\inf B = \sqrt{2}$, assume by way of contradiction that

$$\sqrt{2} < \inf B.$$

Then, using the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, we obtain that there exists $q \in \mathbb{Q}$ such that

$$\sqrt{2} < q < \inf B.$$

We then have that

$$2 < q^2,$$

which shows that $q \in B$, since $q > \sqrt{2} > 0$. However, $q < \inf B$ and $q \in B$ yields a contradiction. Consequently, $\inf B = \sqrt{2}$. Applying the result of part (a) in this problem we then obtain that there exists a sequence, $\{q_n\}$, of numbers in B which converges to $\sqrt{2}$. $\square$