## Review Problems for Exam #2

1. Suppose that the sequence $(x_n)$ converges to $a \neq 0$, where $x_n \neq 0$ for all $n \in \mathbb{N}$. Prove that the sequence $\left(\dfrac{1}{x_n}\right)$ converges to $\dfrac{1}{a}$.

2. Let $(x_n)$ denote a sequence that converges to $x$. Prove that for any $m \in \mathbb{N}$,
$$\lim_{n\to\infty} x_n^m = x^m.$$

3. Let $\delta > 0$ and define $y_n = \dfrac{1}{(1+\delta)^n}$ for all $n \in \mathbb{N}$.
   (a) Use the estimate $(1+\delta)^n > n\delta$, for all $n \in \mathbb{N}$, to prove that the sequence $(y_n)$ converges to 0.
   (b) Define $x_n = x^n$. Prove that if $|x| < 1$, then $(x_n)$ converges. What is $\lim_{n\to\infty} x_n$?

4. Let $(x_n)$ denote a sequence of real numbers.
   (a) Prove that if $(x_n)$ converges then $(x_n^2)$ converges.
   (b) Show that the converse of the statement in part (a) is not true.

5. Let $x$, $a$ and $b$ denote a real numbers.
   (a) Derive the factorization: $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$. *Suggestion:* Let $S = 1 + x + x^2 + \cdots + x^{n-2} + x^{n-1}$ and compute $xS$ and $xS - S$.
   (b) Derive the factorization formula
$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}n + a^{n-3}b^2 + \cdots + b^{n-1})$$
   (c) Let $a$ and $b$ denote positive real numbers, and $n$ a natural number. Prove that
$$a > b \text{ if and only if } a^n > b^n.$$

6. Given $a > 0$ and $n \in \mathbb{N}$, prove that there exists a unique positive solution to the equation $x^n = a$.

   *Note:* In this problem, you might need to use the binomial expansion
$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}, \text{ where } \binom{n}{k} = \frac{n!}{k!(n-k)!}, \text{ for } k = 0, 1, 2, \ldots, n.$$

7. Let $a$ and $b$ denote positive real numbers. For each natural number $n$, let $a^{1/n}$ denote the unique positive solution to the equation $x^n = a$.
   (a) Prove that if $b \leqslant 1$, then $b^m \leqslant 1$ for all $m \in \mathbb{N}$.
   (b) Show that if $a > 1$, then $a^{1/n} > 1$ for all $n \in \mathbb{N}$.
   (c) Prove that if $a > 1$, then $a^{m/n} > 1$ for all $m, n \in \mathbb{N}$, where $a^{m/n} = (a^{1/n})^m$.

8. Let $a$ and $b$ denote positive real, and $n$ a natural number. Prove that
$$a > b \text{ if and only if } a^{1/n} > b^{1/n}.$$

9. Let $a$ denote a positive real number.
   (a) Show that if $a > 1$, then $a - 1 > n(a^{1/n} - 1)$ for all $n \in \mathbb{N}$. Deduce that $\lim_{n\to\infty} a^{1/n} = 1$, for $a > 1$.
   (b) Prove that for any positive real number $a$, $\lim_{n\to\infty} a^{1/n} = 1$.

10. Define $x_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}$ for $n \in \mathbb{N}$.
   (a) Multiply the expression for $x_n$ by $1/2$ and obtain that $x_n = 2 - \frac{2}{2^n}$ for $n \in \mathbb{N}$.
   (b) Deduce that $(x_n)$ converges to 2.

11. Define $s_n = \sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!}$ for all $n = 1, 2, 3, \ldots$
   (a) Show that $s_n \leqslant 1 + x_n$, where $x_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}$, for all $n = 1, 2, 3, \ldots$
   (b) Show that the sequence $(s_n)$ is increasing and bounded and, therefore, it converges.
   (c) Denote the limit of $(s_n)$ by $e$ and show that $2.5 \leqslant e \leqslant 3$.