Solutions to Review Problems for Exam #2

1. Suppose that the sequence (x_n) converges to $a \neq 0$, where $x_n \neq 0$ for all $n \in \mathbb{N}$. Prove that the sequence $\left(\dfrac{1}{x_n}\right)$ converges to $\dfrac{1}{a}$.

   Proof: Assume $\lim\limits_{n\to\infty} x_n = a$, where $a \neq 0$. Then, there exists $N_1 \in \mathbb{N}$ such that

   $$n \geqslant N_1 \Rightarrow |x_n - a| < \frac{|a|}{2}.$$

   It then follows by the triangle inequality that

   $$n \geqslant N_1 \Rightarrow |x_n| > \frac{|a|}{2}.$$

   Thus, for $n \geqslant N_1$,

   $$\left|\frac{1}{x_n} - \frac{1}{a}\right| = \frac{|x_n - a|}{|a||x_n|} < \frac{2}{|a|^2}|x_n - a|.$$

   It then follows by the Squeeze Theorem for sequences that

   $$\lim_{n\to\infty}\left|\frac{1}{x_n} - \frac{1}{a}\right| = 0,$$

   since $\lim\limits_{n\to\infty}|x_n - a| = 0$. Consequently, $\left(\dfrac{1}{x_n}\right)$ converges to $\dfrac{1}{a}$. □

2. Let (x_n) denote a sequence that converges to x. Prove that for any $m \in \mathbb{N}$,

   $$\lim_{n\to\infty} x_n^m = x^m.$$

   Proof: We use induction on $m \in \mathbb{N}$. The case $m = 1$ is true by the assumption that (x_n) converges to x.

   Next, assume that $\lim\limits_{n\to\infty} x_n^m = x^m$, and write

   $$x_n^{m+1} = x_n \cdot x_n^m.$$

   Thus, x_n^{m+1} is the product of two convergent sequences by the inductive hypothesis. We then have that

   $$\lim_{n\to\infty} x_n^{m+1} = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} x_n^m = x \cdot x^m = x^{m+1}.$$

   This completes the induction argument. □

3. Let $\delta > 0$ and define $y_n = \dfrac{1}{(1+\delta)^n}$ for all $n \in \mathbb{N}$.

   (a) Use the estimate $(1+\delta)^n > n\delta$, for all $n \in \mathbb{N}$, to prove that the sequence (y_n) converges to 0.

   Solution: From $(1+\delta)^n > n\delta$, for all $n \in \mathbb{N}$, we obtain that

   $$0 < y_n < \frac{1}{\delta n} \quad \text{for all } n \in \mathbb{N}.$$

   It then follows by the Squeeze Theorem for sequences that (y_n) converges to 0. □

   (b) Define $x_n = x^n$. Prove that if $|x| < 1$, then (x_n) converges. What is $\lim\limits_{n\to\infty} x_n$?

   Solution: We show that $\lim\limits_{n\to\infty} |x_n| = 0$. This will imply that (x_n) converges to 0 if $|x| < 1$.
   Observe that

   $$\begin{aligned} |x_n| &= |x|^n \\ &= \frac{1}{\left(\dfrac{1}{|x|}\right)^n} \\ &= \frac{1}{(1+\delta)^n}, \end{aligned}$$

   where $\delta = \dfrac{1}{|x|} - 1 > 0$, since $|x| < 1$. It then follows by part (a) that

   $$\lim_{n\to\infty} |x_n| = \lim_{n\to\infty} \frac{1}{(1+\delta)^n} = 0.$$

   □

4. Let (x_n) denote a sequence of real numbers.

   (a) Prove that if (x_n) converges then (x_n^2) converges.

   Proof: Observe that $x_n^2 = x_n \cdot x_n$. Consequently, if (x_n) converges to $x \in \mathbb{R}$, then (x_n^2) converges to x^2. □

   (b) Show that the converse of the statement in part (a) is not true.

Solution: Take $x_n = (-1)^n$ for all $n \in \mathbb{N}$. Then, $x_n^2 = 1$ for all $n \in \mathbb{N}$. Thus, (x_n^2) converges, but (x_n) does not. □

5. Let x, a and b denote a real numbers.

   (a) Derive the factorization: $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$. *Suggestion:* Let $S = 1 + x + x^2 + \cdots + x^{n-2} + x^{n-1}$ and compute xS and $xS - S$.

   Solution: Compute
$$xS = x + x^2 + \cdots + x^{n-1} + x^n = S - 1 + x^n.$$
It then follows that
$$xS - S = x^n - 1,$$
from which we get that
$$x^n - 1 = (x-1)S = (x-1)(1 + x + x^2 + \cdots + x^{n-2} + x^{n-1}).$$
□

   (b) Derive the factorization formula
$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}n + a^{n-3}b^2 + \cdots + b^{n-1})$$

   Solution: If $b = 0$, there is nothing to prove since $a^n = aa^{n-1}$. Thus, assume that $b \neq 0$ and write
$$\begin{aligned} a^n - b^n &= b^n\left[\left(\frac{a}{b}\right)^n - 1\right] \\ &= b^n(x^n - 1), \end{aligned}$$
where we have set $x = \dfrac{a}{b}$. Thus, using the factorization formula derived in part (a),
$$\begin{aligned} a^n - b^n &= b^n(x-1)(1 + x + x^2 + \cdots + x^{n-2} + x^{n-1}) \\ &= b^n\left(\frac{a}{b} - 1\right)\left(1 + \frac{a}{b} + \left(\frac{a}{b}\right)^2 + \cdots + \left(\frac{a}{b}\right)^{n-2} + \left(\frac{a}{b}\right)^{n-1}\right) \\ &= (a-b)b^{n-1}\left(1 + \frac{a}{b} + \frac{a^2}{b^2} + \cdots + \frac{a^{n-2}}{b^{n-2}} + \frac{a^{n-1}}{b^{n-1}}\right) \\ &= (a-b)\left(b^{n-1} + ab^{n-2} + a^2b^{n-3} + \cdots + a^{n-2}b + a^{n-1}\right), \end{aligned}$$
which was to be shown. □

(c) Let a and b denote positive real numbers, and n a natural number. Prove that

$$a > b \text{ if and only if } a^n > b^n.$$

Solution: Assume that $a > b$; then $a - b > 0$. It then follows that

$$a^n - b^n = (a-b)(b^{n-1} + ab^{n-2} + a^2b^{n-3} + \cdots + a^{n-2}b + a^{n-1}) > 0,$$

since a and b are positive. Thus, $a^n > b^n$.
Conversely, assume that $a^n > b^n$. Then, $a^n - b^n > 0$. Thus,

$$(b^{n-1} + ab^{n-2} + a^2b^{n-3} + \cdots + a^{n-2}b + a^{n-1})(a-b) > 0.$$

Multiplying by the multiplicative inverse of $b^{n-1}+ab^{n-2}+a^2b^{n-3}+\cdots + a^{n-2}b + a^{n-1}$, which exists and is positive because a and b are positive, we obtain that

$$a - b > 0,$$

which implies that $a > b$. □

6. Given $a > 0$ and $n \in \mathbb{N}$, prove that there exists a unique positive solution to the equation $x^n = a$.

Note: In this problem, you might need to use the binomial expansion

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}, \text{ where } \binom{n}{k} = \frac{n!}{k!(n-k)!}, \text{ for } k = 0, 1, 2, \ldots, n.$$

Solution: Suppose first that $a > 1$. (Note that if $a = 1$, the $x = 1$ solves $x^n = a$).
Define $A = \{t \in \mathbb{R} \mid t > 0 \text{ and } t^n \leqslant a\}$. Then, for the case $a > 1$, $A \neq \emptyset$ since $1 \in A$, because $1 = 1^n < a$. Next, we see that A is bounded. This follows from the fact that $a < a^n$ for all $n \in \mathbb{N}$ since $a > 1$. It then follows that $t \in A$ implies that $t > 0$ and

$$t^n < a < a^n,$$

from which we get that $t < a$, and therefore a is an upper bound for A. Thus, the supremum of A exists. Let $s = \sup(A)$. We show that $s^n = a$. For each $k \in \mathbb{N}$, there exists $t_k \in A$ such that

$$s - \frac{1}{k} < t_k \leqslant s.$$

It then follows that

$$\lim_{k\to\infty} t_k = s.$$

Consequently,

$$\lim_{k\to\infty} t_k^n = s^n,$$

which implies that $s^n \leqslant a$, since $t_k^n \leqslant a$ for all $k \in \mathbb{N}$.
Suppose, by way of contradiction, that $s^n < a$. Then, $a - s^n > 0$ and therefore

$$\frac{a - s^n}{\displaystyle\sum_{k=1}^{n} \binom{n}{k} s^k} > 0.$$

Then, there exists an integer $m > 1$ such that

$$\frac{1}{m} < \frac{a - s^n}{\displaystyle\sum_{k=1}^{n} \binom{n}{k} s^k}.$$

Put $\gamma = \dfrac{1}{m}$; then $0 < \gamma < 1$ and

$$\gamma \left(\sum_{k=1}^{n} \binom{n}{k} s^k \right) < a - s^n. \tag{1}$$

By the binomial expansion theorem,

$$\begin{aligned} (s+\gamma)^n &= s^n + \sum_{k=1}^{n} \binom{n}{k} s^k \gamma^{n-k} \\ &< s^n + \gamma \left(\sum_{k=1}^{n} \binom{n}{k} s^k \right), \end{aligned}$$

since $\gamma < 1$. In then follows from the estimate in (1) that

$$(s+\gamma)^n < a,$$

which shows that $s+\gamma \in A$, which is a contradiction since $s = \sup(A)$. Consequently, $s^n = a$. Thus, $x^n = a$ has a positive solution for the case $a > 1$.

To show that there is at most one solution to $x^n = a$. Suppose that there exist positive, real numbers, s_1 and s_2, such that $s_1^n = a$ and $s_2^n = a$. It then follows that

$$0 = s_1^n - s_2^n = (s_1 - s_2)(s_1^{n-1} + s_1^{n-2}s_2 + \cdots s_2^{n-1}),$$

from which we obtain that $s_1 - s_2 = 0$, which implies that $s_1 = s_2$.

Finally, observe that if $0 < a < 1$, then $\dfrac{1}{a} > 1$; so, by what we have just proved, there exists a unique $y \in \mathbb{R}$ with $y^n = \dfrac{1}{a}$. Then $\dfrac{1}{y^n} = a$, or $\left(\dfrac{1}{y}\right)^n = a$. Thus, $x = \dfrac{1}{y}$ solves $x^n = a$. □

7. Let a and b denote positive real numbers. For each natural number n, let $a^{1/n}$ denote the unique positive solution to the equation $x^n = a$.

   (a) Prove that if $b \leqslant 1$, then $b^m \leqslant 1$ for all $m \in \mathbb{N}$.

   Solution: Suppose that $b \leqslant 1$. We prove that $b^m \leqslant 1$ for all $m \in \mathbb{N}$ by induction on m.
   For $m = 1$, the result follows by the assumption that $b \leqslant 1$.
   Suppose that $b^m \leqslant 1$ and consider

   $$b^{m+1} = b^m \cdot b \leqslant (1) \cdot (1) = 1.$$

   □

   (b) Show that if $a > 1$, then $a^{1/n} > 1$ for all $n \in \mathbb{N}$.

   Solution: Suppose that $a > 1$. We prove that $a^{1/n} > 1$ by contradiction. Thus, suppose that $a^{1/n} \leqslant 1$. Then, by the result of the previous part,

   $$(a^{1/n})^n \leqslant 1,$$

   from which we get that $a \leqslant 1$, which contradicts the hypothesis that $a > 1$. Hence, $a > 1$ implies that $a^{1/n} > 1$. □

   (c) Prove that if $a > 1$, then $a^{m/n} > 1$ for all $m, n \in \mathbb{N}$, where $a^{m/n} = (a^{1/n})^m$.

   Solution: Suppose that $a > 1$. It then follows from part (b) that $a^{1/n} > 1$. Consequently, $(a^{1/n})^m > 1$, which can be proved by an induction argument like the one used in part (a). It then follows that

   $$a^{m/n} > 1.$$

   □

8. Let a and b denote positive real, and n a natural number. Prove that

$$a > b \text{ if and only if } a^{1/n} > b^{1/n}.$$

Proof: Let $a^{1/n}$ and $b^{1/n}$ be the unique positive solutions to the equations $x^n = a$ and $x^n = b$, respectively. Then, $(a^{1/n})^n = a$ and $(b^{1/n})^n = b$. By the result of part (c) of Problem 5,

$$a^{1/n} > b^{1/n} \text{ if and only if } (a^{1/n})^n > (b^{1/n})^n,$$

from which we get that

$$a^{1/n} > b^{1/n} \text{ if and only if } a > b.$$

□

9. Let a denote a positive real number.

(a) Show that if $a > 1$, then $a - 1 > n(a^{1/n} - 1)$ for all $n \in \mathbb{N}$. Deduce that $\lim_{n\to\infty} a^{1/n} = 1$, for $a > 1$.

Solution: Suppose that $a > 1$ and compute

$$a-1 = (a^{1/n})^n - 1 = (a^{1/n} - 1)(a^{(n-1)/n} + a^{(n-2)/n} + \cdots + a^{1/n} + 1).$$

Then using the result of part (c) of Problem 7, we get that

$$a - 1 > (a^{1/n} - 1) \cdot n,$$

which was to be shown.
It then follows that

$$0 < a^{1/n} - 1 < \frac{a-1}{n} \quad \text{for all } n \in \mathbb{N}.$$

Consequently, by the Squeeze Theorem for sequences,

$$\lim_{n\to\infty} a^{1/n} = 1.$$

□

(b) Prove that for any positive real number a, $\lim_{n\to\infty} a^{1/n} = 1$.

Solution: Let $a > 0$. Then, $a > 1$, $a = 1$ or $0 < a < 1$. If $a > 1$, then result follows by part (a). If $a = 1$ the $a^{1/n} = 1$ for all $n \in \mathbb{N}$ and so the result also holds true in this case. Thus, it remains to consider the case $0 < a < 1$.
If $0 < a < 1$, then $\dfrac{1}{a} > 1$, and so, by part (a),

$$\lim_{n\to\infty} \left(\frac{1}{a}\right)^{1/n} = 1.$$

It then follows that

$$\lim_{n\to\infty} \frac{1}{a^{1/n}} = 1,$$

from which we obtain that

$$\lim_{n\to\infty} a^{1/n} = \lim_{n\to\infty} \frac{1}{\dfrac{1}{a^{1/n}}} = \frac{1}{\displaystyle\lim_{n\to\infty} \frac{1}{a^{1/n}}} = 1.$$

□

10. Define $x_n = 1 + \dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \cdots + \dfrac{1}{2^{n-1}}$ for $n \in \mathbb{N}$.

(a) Multiply the expression for x_n by 1/2 and obtain that $x_n = 2 - \dfrac{2}{2^n}$ for $n \in \mathbb{N}$.

Solution: Compute

$$\begin{aligned}
\frac{1}{2}x_n &= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^n} \\
&= 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^n} - 1 \\
&= x_n + \frac{1}{2^n} - 1.
\end{aligned}$$

It then follows that

$$\frac{1}{2}x_n = 1 - \frac{1}{2^n},$$

from which we get that

$$x_n = 2 - \frac{2}{2^n}.$$

□

(b) Deduce that (x_n) converges to 2.

Solution: Since $\lim\limits_{n\to\infty} \dfrac{1}{2^n} = 0$, we get that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \left(2 - \frac{2}{2^n}\right) = 2.$$

□

11. Define $s_n = \displaystyle\sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!}$ for all $n = 1, 2, 3, \ldots$

(a) Show that $s_n \leqslant 1 + x_n$, where $x_n = 1 + \dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \cdots + \dfrac{1}{2^{n-1}}$, for all $n = 1, 2, 3, \ldots$

Solution: We first show that $k! \geqslant 2^{k-1}$, for all $k \in \mathbb{N}$, by induction on k.
For $k = 1$ we obtain $k! = 1$ and $2^{k-1} = 2^0 = 1$; so the result holds true in this case.
Next, we assume that $k! \geqslant 2^{k-1}$ and seek to prove that $(k+1)! \geqslant 2^k$.
Compute $(k+1)! = (k+1) \cdot k! \geqslant (k+1)2^{k-1}$, by the inductive hypothesis; so that, since $k \geqslant 1$,

$$(k+1)! = (k+1) \cdot k! \geqslant 2 \cdot 2^{k-1} = 2^k,$$

which was to be shown.
Next, use the estimate we just proved to get

$$s_n = \sum_{k=0}^{n} \frac{1}{k!} = 1 + \sum_{k=1}^{n} \frac{1}{k!} \leqslant 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} = 1 + x_n.$$

□

(b) Show that the sequence (s_n) is increasing and bounded and, therefore, it converges.

Solution: Thus, by the result of part (a) in this problem and part (a) in Problem 10,

$$s_n \leqslant 3 - \frac{2}{2^n} < 3 \quad \text{for all } n \in \mathbb{N}.$$

Thus, $0 < s_n < 3$ for all n, and therefore (s_n) is bounded.
The sequence (s_n) is also increasing, since

$$s_{n+1} = s_n + \frac{1}{(n+1)!} > s_n \quad \text{for all } n = 1, 2, 3, \ldots$$

Consequently, (s_n) is increasing and bounded and therefore (s_n) converges. □

(c) Denote the limit of (s_n) by e and show that $2.5 \leqslant e \leqslant 3$.

Solution: Observe that

$$1 + 1 + \frac{1}{2} < s_n < 1 + x_n$$

for all $n \geqslant 3$. Thus, taking the limit as $n \to \infty$,

$$2.5 \leqslant e \leqslant 3,$$

since $\lim\limits_{n\to\infty} x_n = 2$. □