Handout #2: The Real Numbers System Axioms

I. Field Axioms

The set of real numbers $\mathbb{R}$ has two algebraic operations: **addition** (the sum of any two elements x and y of $\mathbb{R}$ being denoted by $x + y$) and **multiplication** (the product of any two elements x and y of $\mathbb{R}$ being denoted by xy) defined for any pair of elements in the set. These operations satisfy the properties of a **field**, which are the following:

Closure properties

(F_1) For any two real numbers x and y, $x + y$ and xy are real numbers.

Properties of addition

(F_2) (*Commutativity*). For any x and y in $\mathbb{R}$,

$$x + y = y + x.$$

(F_3) (*Associativity*). For any three elements x, y, and z in $\mathbb{R}$,

$$(x + y) + z = x + (y + z).$$

(F_4) (*Existence of an additive identity*). There exists a real number 0 with the property:

$$x + 0 = x \quad \text{for all } x \text{ in } \mathbb{R}.$$

(F_5) (*Existence of additive inverses*). For every x in $\mathbb{R}$, there exists y in $\mathbb{R}$ with the property:

$$x + y = 0.$$

Properties of multiplication

(F_6) (*Commutativity*). For any pair of real numbers x and y,

$$xy = yx.$$

(F_7) (*Associativity*). For any three elements x, y, and z in $\mathbb{R}$,

$$(xy)z = x(yz).$$

(F_8) (*Existence of an multiplicative identity*). There exists a real number 1 such that $1 \neq 0$ and

$$x \cdot 1 = x \quad \text{for all } x \text{ in } \mathbb{R}.$$

(F_9) (*Existence of multiplicative inverses for non–zero real numbers*). For every x in $\mathbb{R}$ such that $x \neq 0$, there exists $y$in $\mathbb{R}$ such that

$$xy = 1.$$

Distributive property

(F_{10}) For any real numbers x, y and z,

$$x(y + z) = xy + xz.$$

II. **Order Axioms**

We designate a certain subset P of $\mathbb{R}$ as the "positive numbers" in $\mathbb{R}$. This set P is "invariant" under the operations in $\mathbb{R}$; i.e., if x and y are in P, then $x + y$ and xy are also in P. The set P induces an **order relation** in $\mathbb{R}$ as follows: we say that $x < y$ if $y - x \in P$. The notation $x \leq y$ means $x < y$ or $x = y$. Similarly, we define $x > y$ to mean $x - y \in P$, and $x \geq y$ to mean $x > y$ or $x = y$.

The field $\mathbb{R}$ is an **ordered field** since the following properties hold:

(O_1) (*Trichotomy property*). If $x \in \mathbb{R}$, then $x = 0$ or $x > 0$ or $x < 0$. (Note: only one of these three possibilities can hold.)

(O_2) If $x > 0$ and $y > 0$, then $x + y > 0$.

(O_3) If $x > 0$ and $y > 0$, then $xy > 0$.

III. **Completeness Axiom**

Let A be a subset of $\mathbb{R}$. We say that b is an **upper bound** for A if $x \leq b$ for all $x \in A$. A number c is called a **least upper bound** for A if c is an upper bound for A and $c \leq b$ for any upper bound b for A. The ordered field $\mathbb{R}$ is said to be **complete** since it satisfies the following

(C) (*Least upper bound property*). Every non-empty subset of $\mathbb{R}$ that has an upper bound has a least upper bound.

Remarks

1. Given $x \in \mathbb{R}$, the additive inverse for x given by the field axiom (F_5) is unique and is denoted by $-x$. The expression $y - x$, for any pair of real numbers x and y, is then interpreted as $y + (-x)$.

2. Given a non–zero real number x, the multiplicative inverse for x given by the field axiom (F_9) is unique and is denoted by x^{-1} or $\dfrac{1}{x}$. The expression $\dfrac{y}{x}$, for $x, y \in \mathbb{R}$ with $x \neq 0$, is then interpreted as yx^{-1} or $y\dfrac{1}{x}$.

3. The set of rational numbers $\mathbb{Q}$ is a sub-field of $\mathbb{R}$; that is, the field axioms (F_1)–(F_{10}) hold true for $\mathbb{Q}$ as well. The rational numbers are also an ordered field with the same order relation defined in $\mathbb{R}$. However, $\mathbb{Q}$ is not a complete field.