Problem Set #1: The Set of Real Numbers

1. Let n be a natural number. Show that if n is even, then so is n^2. Conversely, show that if n^2 is even, then n is even.

2. Let n be a natural number. Show that n is a multiple of 3 if and only if n^2 is a multiple of 3.

3. Show that $\sqrt{3}$ is irrational.

4. Show that $\sqrt{6}$ is irrational.

5. *Ture or false.*

   (a) If α and β are irrational, then $\alpha + \beta$ is irrational.

   (b) If α and β are irrational, then $\alpha\beta$ is irrational.

6. The following are consequences of the field axioms for the real numbers:

   (a) (*The cancelation laws*)

      i. Let x, y and z be real numbers. If $x + z = y + z$, then $x = y$.

      ii. Let x, y and z be real numbers with $z \neq 0$. If $xz = yz$, then $x = y$.

   (b) Show that 0 and 1 are unique.

   (c) Given $x \in \mathbb{R}$, $-x$ and x^{-1} are unique.

   (d) $x \cdot 0 = 0$ for all $x \in \mathbb{R}$.

   (e) $(-1) \cdot (-x) = x$ for all $x \in \mathbb{R}$, where $-x$ is the unique additive inverse of x given by the field axiom (F_5).

   (f) Let a, b and c be real numbers with $a \neq 0$, then the equation $ax + b = c$ has a unique solution.

7. Let a and b be real numbers. If $ab = 0$, then either $a = 0$ or $b = 0$.

8. Show that the set of rational numbers $\mathbb{Q}$ is a sub–field of the set of real numbers.

9. The following are consequences of the field and order axioms for the real numbers. Let x, y and z be real numbers.

   (a) If $x < y$ and $y < z$, then $x < z$.

   (b) If $x < y$, then $x + z < y + z$.

   (c) If $x < y$ and $z > 0$, then $xz < yz$.

   (d) If $x < 0$ and $y < 0$, then $xy > 0$.

   (e) If $x < y$ and $z < 0$, then $yz < xz$.

   (f) If $x < y$, then $-y < -x$.

   (g) $x^2 \geq 0$ for any real number x.

   (h) $1 > 0$

   (i) If $x > 0$, then $x^{-1} > 0$.

   (j) If $0 < x < y$, then $0 < \dfrac{1}{y} < \dfrac{1}{x}$.

10. For any $x \in \mathbb{R}$, $x \leq |x|$.

11. Show that $\mathbb{Q}$ is an ordered subfield of $\mathbb{R}$.