Solutions to Assignment #3

1. *[Problem 1.1.11 on page 8 in Allman and Rhodes]* Explain why the model

$$\Delta P = rP$$

leads to the formula

$$P_t = (1+r)^t P_o.$$

Solution: Write the equation in the form

$$P_{t+1} = (1+r)P_t, \tag{1}$$

and assume that P_t is P_o when $t = 0$. Then, by (1),

$$P_1 = (1+r)P_o,$$

and therefore, using (1) again (this time for $t = 1$),

$$P_2 = (1+r)P_1 = (1+r)1+r)P_o = (1+r)^2 P_o.$$

We may now proceed by induction on n. Assume therefore that we have established that

$$P_n = (1+r)^n P_o.$$

Then, applying (1) with $t = n$,

$$P_{n+1} = (1+r)P_n = (1+r)(1+r)^n P_o = (1+r)^{n+1} P_o.$$

Hence, by the Principle of Mathematical Induction,

$$P_t = (1+r)^t P_o \quad \text{for} \quad t = 0, 1, 2, \ldots \quad \square$$

2. *[Problem 1.2.7 on page 18 in Allman and Rhodes]*

Solution: Figure 1 shows the plot of the insect population data displayed in Table 1.6, page 18, of Allman and Rhodes. It shows a typical S–curve associated with the logistic growth model. By inspecting the graph, we can get an estimate for the carrying capacity of the population, K, of about 8.5 (Note: the horizontal line at $N = 8.5$ is also shown in the graph). To estimate r we may take the first two data points and compute ΔN in the equation

$$\Delta N = rN_t\left(1 - \frac{N_t}{K}\right). \tag{2}$$

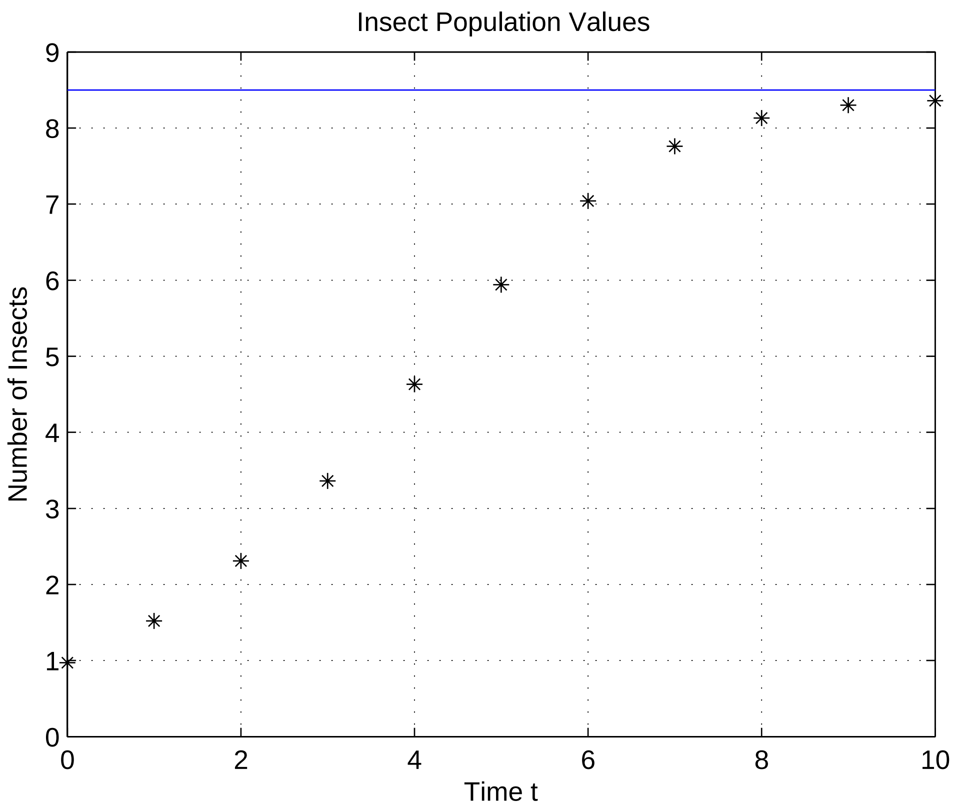

Figure 1: Plot of Insect Population Values in Table 1.6 of Allman and Rhodes, p. 18

Taking $t = 0$, so that $\Delta N = N_1 - N_o = 1.52 - 0.97 = 0.55$, and $K = 8.5$, we can estimate r from the last equation by solving for it

$$r = \frac{\Delta N}{N_o\left(1 - \frac{N_o}{K}\right)} \approx \frac{0.55}{0.97\left(1 - \frac{0.97}{8.5}\right)} \approx 0.64. \quad \square$$

Notice that this estimate for r only takes into account the first two data points so we don't expect it to be be an estimate that works well for the entire data set. The estimate for K is essentially a graphical estimate and we don't have a handle on how accurate it is.

There is a better way to estimate r and K which takes into account the whole set of data and which takes advantage of the computational capabilities of MATLAB®. The idea is to write the logistic equation in (2) in the form

$$\frac{\Delta N}{N} = r\left(1 - \frac{N}{K}\right),$$

or

$$\frac{\Delta N}{N} = r - \frac{r}{K}N, \tag{3}$$

and observe that equation (3) says that there is a linear relation between the per–capita growth rate, $\dfrac{\Delta N}{N}$, and N. Thus, if we plot the values of $\dfrac{\Delta N}{N}$ given by the data versus N, the points should arrange themselves close to a straight line with slope $m = -\dfrac{r}{K}$ and y–intercept r. Hence, plotting the per–capita growth rates, fitting a straight line through the data points, and computing the slope and y–intercept of the resulting line should give us estimates for r and K.

Figure 2 on page 4 shows the graph of $\dfrac{\Delta N}{N}$ versus N obtained using MATLAB®. Observe that the data points fall suspiciously into one, neat straight line (an indication that the data in Table 1.6 on page 18 of Allman and Rhodes, most likely, is made up; that is, generated by the logistic model and not from an actual experiment). We can trace the line through the points as shown in Figure 3 on page 4. We can then estimate the y–intercept and the slope of the line graphically, or we can use the `polyfit` function in MATLAB®to obtain the slope and y intercept of the least–square regression line. Letting `Y` be an array containing the values for $\dfrac{\Delta N}{N}$ and `X` the array of values of `N(i)`, for `i = 1, 2, ..., 10`, and typing

```
polyfit(X,Y,1)
```

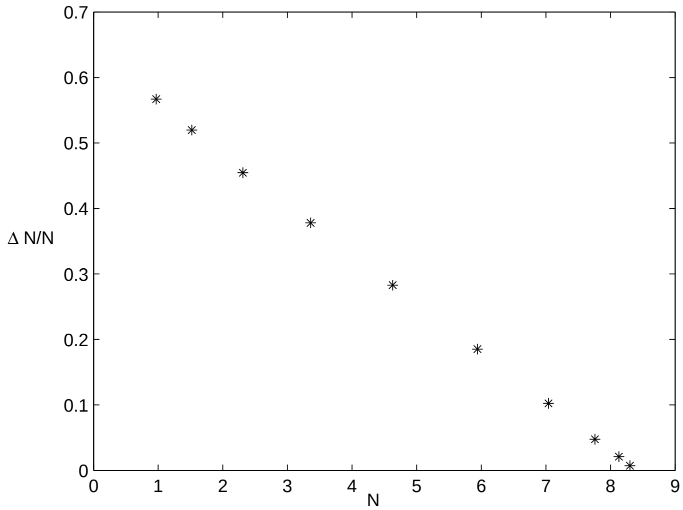

Figure 2: Plot of $\dfrac{\Delta N}{N}$ versus N

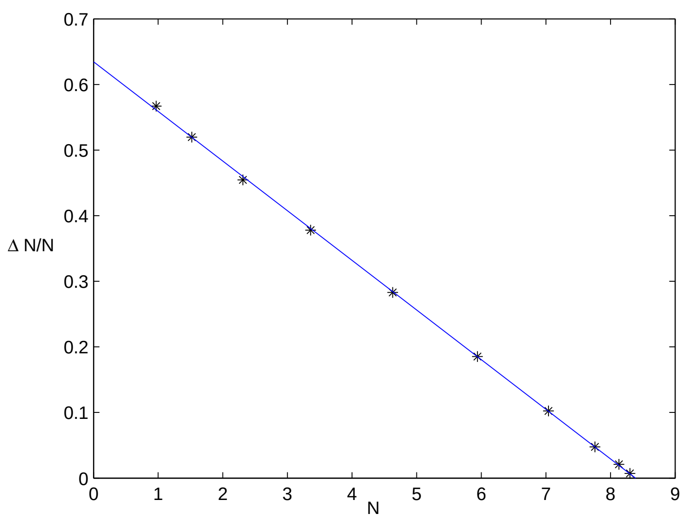

Figure 3: Plot of $\dfrac{\Delta N}{N}$ versus N and best–fit line

yields

```
m        y-int
```

the slope, m, and the y–intercept of the best–fitting line through the data points. In this particular case, we obtain the values

```
-0.0756    0.6344
```

Thus,

$$m = -0.0756 \quad \text{and} \quad r = 0.6344.$$

Using the expression $m = -\dfrac{r}{K}$ for the slope, we obtain an estimate for the carrying capacity, K, to be

$$K = -\frac{r}{m} \approx 8.3883.$$

3. *[Problem 1.2.8 on page 18 in Allman and Rhodes]* Suppose the growth of a population is modeled by the equation

$$N_{t+1} = N_t + 0.2N_t\left(1 - \frac{N_t}{200000}\right), \tag{4}$$

where N_t is measured in *individuals.*

(a) Find an equation of the same form, describing the same model, but with the population measured in *thousands of individuals.*

Solution: Let $M_t = \dfrac{1}{1000}N_t$ for each $t = 0, 1, 2, \ldots$ Divide equation (**??**) by 1000 and rearrange the last term to get

$$\frac{N_{t+1}}{1000} = \frac{N_t}{1000} + 0.2\frac{N_t}{1000}\left(1 - \frac{N_t/1000}{200}\right).$$

Thus, the equation for M_t reads

$$M_{t+1} = M_t + 0.2M_t\left(1 - \frac{M_t}{200}\right). \quad \square$$

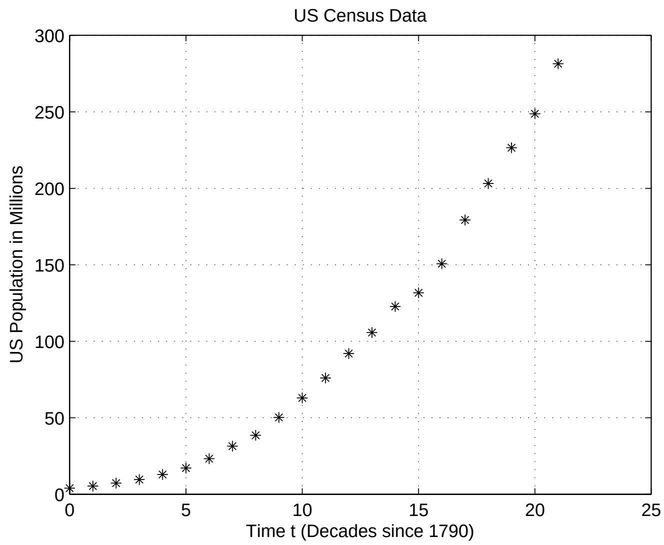

Figure 4: US Census Data since 1790

(b) Find the equation of the same form, describing the same model, but with the population measured in units chosen so that the carrying capacity is 1 in those units

Solution: This time we let $M_t = \dfrac{1}{200000} N_t$; that is, we divide N_t by the carrying capacity. Then, proceeding as in the previous example (this time dividing by 200000) we obtain

$$M_{t+1} = M_t + 0.2M_t\,(1 - M_t)\,. \quad \square$$

4. *(US Census Data.)* The MS Excel file CensusDataUS in the Math 36 webpage (see the courses website at `http://pages.pomona.edu/~ajr04747`) contains the total US population (in millions of people) for each year that a census has been taken in the United States.

(a) Use MATLAB® to get a plot of the US population as a function of t, where t is in units of 10 years since the year 1790.

Solution: Figure 4 on page 6 shows the plot. $\square$

(b) If the US population follows a Malthusian model, what would the growth rate λ be? Using this value of λ, compute the population values that the model predicts for $t = 1, 2, 3, \ldots$ Plot the predicted and actual values on the same graph. How well do these predictions compare with the actual data?

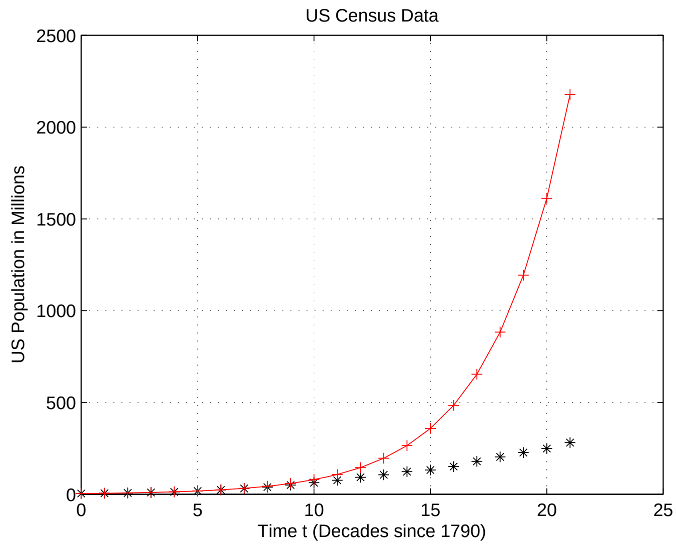

Figure 5: US Census Data and Values Predicted by Malthusian Model

Solution: If we let N_t denote the US population (in millions) for $t = 0, 1, 2, \ldots$, the Malthusian model predicts that $N_t = N_o\lambda^t$ for $t = 0, 1, 2, \ldots$ We can take N_o to be the US population in 1790; i.e., $N_o = 3.929$. We can estimate λ by computing

$$\lambda = \frac{N_1}{N_o} \approx \frac{5.308}{3.929} \approx 1.351.$$

We use these values of N_o and λ to compute the values predicted by the Malthusian model. These values are plotted in Figure 4 (solid curve with '+' for the predicted values) along with the actual US Census data. We can see that the predicted values diverge a great deal from the actual census data. $\square$

5. *(US Census Data, continued).* Starting with the solution to the Malthusian model: $N_t = N_0\lambda^t$, take logarithms on both sides to get

$$\ln N_t = \ln N_0 + t\ln(\lambda).$$

Thus, the relationship between $\ln N_t$ and t should be linear with slope $\ln(\lambda)$ and y–intercept $\ln N_0$.

(a) If `X` represents a row of values, and `Y` another row of values of the same size, the MATLAB® function `polyfit(X,Y,1)` returns the slope m and y–intercept y_o of the line that best fits the data (in the sense of least squares

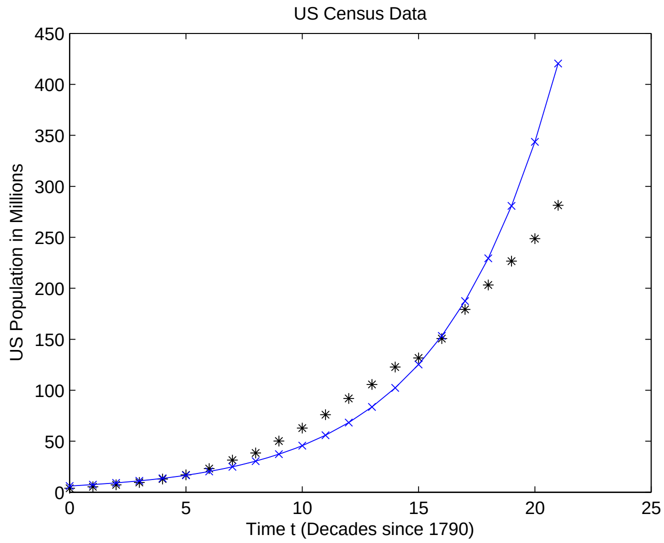

Figure 6: US Census Data and Values Predicted by Malthusian Model

regression) in `X` and `Y`:

$$y = mx + y_o.$$

Use this function to obtain estimates for the values of $\ln N_0$ and $\ln(\lambda)$

Solution: Define `t = [0:21]` and `Y = log(USpop)` in MATLAB®, where `USpop` is the array containing the US Census data. Then, `polyfit(t,Y,1)` returns the slope `m` and y–intercept `b` for the least–squares regression line. The MATLAB®output yields $m = 0.2019$ and $b = 1.8023$. □

(b) Obtain estimates for N_0 and λ, and use them to generate a new set of predicted values for the US population. Plot these, along with the actual data, and assess how good the fit is.

Solution: Using the values for m and b obtained in the previous part, we get the following estimates for λ and N_o: $\lambda = \exp(m) \approx 1.2237$ and $N_o = \exp(b) \approx 6.0634$. As in the previous problem, we can use these values to obtain predicted population values according to the Malthusian model. These values are plotted in Figure 6 (solid curve with 'x' for the predicted values) along with the actual US Census data. We see by inspecting the graph that the fit, though better than the one in the previous problem, is actually not very good.