Solutions to Assignment #4

1. *(Numerical Analysis of the Logistic Equation).* In this problem and the next two, you are asked to use the MATLAB® program `Logistic.m` to explore how the nature of the solutions to the logistic difference equation

$$N_{t+1} = N_t + rN_t(1 - N_t) \tag{1}$$

changes as one varies the parameter r and the initial condition N_o. The code for `Logistic.m` may be found in the Math 36 webpage of the courses website at `http://pages.pomona.edu/~ajr04747`.

Start out with the initial condition $N_o = 0.1$ and consider the following values of r: 1, 1.5, 2, 2.1, 2.25, 2.5 and 2.7. Describe in words the long term behavior of the solution to (1) for each value of r. Is there any significant change in the structure of the solution? Is there anything striking?

Solution: At $r = 1$, the solution appears to level off at $N = 1$ as $t \to \infty$. At $r = 1.5$, solution overshoots the $N = 1$ value, but then oscillates about it with decreasing amplitude until it settles at $N = 1$ for large values of t. At $r = 2$, solution appears to oscillate around $N = 1$ with amplitude apparently decreasing very gradually; it is hard to tell if eventually it will go to $N = 1$. At $r = 2.1$, solutions appears to oscillate about $N = 1$ forever, starting at $t = 3$. At $r = 2.25$, solution is oscillating. At $r = 2.5$, solution oscillates with a different pattern: values appear to repeat after two cycles (period-2 oscillations). At $r = 2.7$, oscillations occur at a pattern that is difficult to predict or describe. □

2. *(Numerical Analysis of the Logistic Equation, continued).* Keep the value of r at 2.7 and try the following initial conditions:

$$N_o = 0.1 \quad \text{and} \quad N_o = 0.101.$$

Before you try the second initial condition, type the MATLAB® command `hold on`. This will allow you to see the plots of the two solutions on the same graph. Is there anything that strikes you? What implications does this result might have on the question of predictability?

Solution: After $t = 8$, the second solution starts to have a very different pattern of oscillation from that of the first solution. After $t = 8$, the solutions have quite different values. This is striking considering that there was not that much difference in the initial conditions (they differ only by 0.001). Thus, varying the initial condition just slightly, generates a very different pattern of oscillation which is hard to predict. □

3. *(Numerical Analysis of the Logistic Equation, continued).*

   (a) What happens when $r = 3$ and t is allowed to range from 0 to 100? How would you describe the solution?

   Solution: Solution seems to oscillate erratically. $\square$

   (b) What happens when $r = 3.01$? Does this result suggest that we need to impose a restriction on r? What should that restriction be?

   Solution: Solution ceases to exist after $t = 10$ (that is, it becomes negative afterwards and is therefore no longer biologically meaningful). Thus, we need to restrict r to lie strictly between 0 and 3; i.e., $0 < r < 3$. $\square$

4. *[Problems 1.1.16 (a)(b) on pages 9 and 10 in Allman and Rhodes]* Suppose the growth of a population is modeled by the difference equation $N_{t+1} = 2N_t$ and the initial condition $N_0 = A$, where t is measured in years.

   (a) Suppose we change the time scale so that one unit in the new scale represents half a year. We express this by the equation $\tau = \dfrac{1}{2}t$. If we let P_τ denote the population size in the new time scale, find the model equation for P_τ so that the growth is still geometric.

Table 1: Changing Time Steps in a Model

t	0		1		2		3
N_t	A		$2A$		$4A$		$8A$
τ	0	1	2	3	4	5	6
P_τ	A	$\sqrt{2}A$	$2A$	$2\sqrt{2}A$	$4A$	$4\sqrt{2}A$	$8A$

Solution: The values in last row in Table 1 were obtained by requiring that P_τ grows geometrically; in particular, we want

$$\frac{P_2}{P_1} = \frac{P_1}{P_0}.$$

Since $P_2 = N_1 = 2A$ and $P_0 = A$, we then get that

$$\frac{2A}{P_1} = \frac{P_1}{A}.$$

Solving for P_1 in the last equation, we obtain $P_1 = \sqrt{2}A$. We then see that every term in that row is obtained by multiplying the previous one by $\sqrt{2}$. Hence

$$P_{\tau+1} = 2^{1/2} P_\tau. \quad \square$$

(b) Produce a new model in which the time scale is given by $\tau = \dfrac{1}{10}t$. Denote the population size by Q_τ.

Solution: As in the previous part, we seek to fill in the quantities Q_i for $i = 1, 2, 3, \ldots$ in the last row of Table 2. In order to do so we need to find a multiplier λ such that $Q_{\tau+1} = \lambda Q_\tau$ for $\tau = 0, 1, 2, 3, \ldots$

Table 2: Time Scale $\tau = t/10$

t	0										1
N_t	A										$2A$
τ	0	1	2	3	4	5	6	7	8	9	10
Q_τ	A	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8	Q_9	$2A$

Suppose for the moment that we have found the multiplier λ, then

$$Q_n = \lambda^n A$$

for all $n = 0, 1, 2, 3, \ldots$. Thus, since $Q_{10} = N_1$, we have that $\lambda^{10} A = 2A$, which implies that $\lambda = 2^{1/10}$. Consequently,

$$Q_{\tau+1} = 2^{1/10} Q_\tau. \quad \square.$$

5. *[Problems 1.1.16 (c)(d) on page 10 in Allman and Rhodes]*

(c) Produce a new model that agrees with N_t at 1–year intervals, but its time scale is given by $\tau = \dfrac{1}{h}t$, for some $h > 0$. Denote the population size by R_τ.

Solution: We seek to find a parameter λ such that $R_{\tau+1} = \lambda R_\tau$ for $\tau = 0, 1, 2, \ldots$ Then, $R_\tau = \lambda^\tau A$ for all τ. In particular, we get that $R_{1/h} = N_1 = 2A$, or $\lambda^{1/h} A = 2A$, which implies that $\lambda^{1/h} = 2$, so that $\lambda = 2^h$. It then follows that

$$R_{\tau+1} = 2^h R_\tau. \quad \square$$

(d) Generalize parts (a)–(c) for the case in which N_t satisfies the difference equation

$$N_{t+1} = kN_t,$$

for some constant k, and we seek to produce a new model that agrees with N_t at 1–year intervals, but its time scale is given by $\tau = \dfrac{1}{h}t$, for some $h > 0$.

Solution: Denote the population size by P_τ, and look for λ such that

$$P_{\tau+1} = \lambda P_\tau$$

for $\tau = 0, 1, 2, \ldots$ Then, $P_\tau = \lambda^\tau A$ for all τ. In particular, we get that $P_{1/h} = N_1 = kA$, or $\lambda^{1/h} A = kA$, which implies that $\lambda^{1/h} = k$, so that $\lambda = k^h$. It then follows that

$$P_{\tau+1} = k^h P_\tau. \quad \square$$