Assignment #6

Due on Friday, February 19, 2010

Read Chapter 4 on *the continuous approach to modeling bacterial growth,* p. 29, in the class lecture notes webpage at `http://pages.pomona.edu/~ajr04747`

Do the following problems

1. Suppose the growth of a population is governed by the *differential equation*

$$\frac{dN}{dt} = -kN$$

where k is a positive constant.

   (a) Explain why this model predicts that the population will decrease as time increases.

   (b) If the population at $t = 0$ is N_o, find the time t, in terms of k, at which the population will be reduced by half.

2. Consider a bacterial population whose relative growth rate is given by

$$\frac{1}{N}\frac{dN}{dt} = K$$

where $K = K(t)$ is a continuous function of time, t.

   (a) Suppose that $N_o = N(0)$ is the initial population density. Verify that

$$N(t) = N_o \exp\left(\int_0^t K(\tau)\ \mathrm{d}\tau\right)$$

   solves the differential equation and satisfies the initial condition.

   (b) Find $N(t)$ if

$$K(t) = \begin{cases} 1 - t & \text{if } 0 \leq t \leq 1; \\ 0 & \text{if } t > 1. \end{cases}$$

   Sketch the graph of $N(t)$

3. For any population (ignoring migration, harvesting, or predation) one can model the relative growth rate by the following conservation principle

$$\frac{1}{N}\frac{dN}{dt} = \text{birth rate (per capita)} - \text{death rate (per capita)} = b - d,$$

where b and d could be functions of time and the population density N.

   (a) Suppose that b and d are linear functions of N given by $b = b_o - \alpha N$ and $d = d_o + \beta N$ where b_o, d_o, α and β are positive constants. Assume that $b_o > d_o$. Sketch the graphs of b and d as functions of N. Give a possible interpretation for these graphs.

   (b) Find the point where the two lines sketched in part (a) intersect. Let K denote the first coordinate of the point of intersection. Show that

$$K = \frac{b_o - d_o}{\alpha + \beta}.$$

   K is the carrying capacity of the population.

   (c) Show that $\dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right)$ where $r = b_o - d_o$ is the intrinsic growth rate.

4. The following equation models the evolution of a population that is being harvested at a constant rate:

$$\frac{dN}{dt} = 2N - 0.01N^2 - 75.$$

Find equilibrium solutions and sketch a few possible solution curves. According to model, what will happen if at time $t = 0$ the initial population densities are 40, 60, 150, or 170?

5. Consider the modified logistic model

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)\left(\frac{N}{T} - 1\right)$$

where $N(t)$ denotes the population density at time t, and $0 < T < K$.

   (a) Find the equilibrium solutions and determine the nature of their stability.

   (b) Sketch other possible solutions to the equation.

   (c) Describe what the model predicts about the population and give a possible explanation.