Solutions to Part II of Exam 1

3. Consider the linear first order differential equation

$$\frac{du}{dt} = au + b,$$

where a and b are real parameters with $a \neq 0$.

(a) Find the equilibrium points of the equation.

Solution: Solve the equation $au + b = 0$ to get that

$$\overline{u} = -\frac{b}{a}$$

is the only equilibrium point since $a \neq 0$. □

(b) Sketch some possible solutions to the equation for the cases $a < 0$ and $a > 0$ in separate graphs. Which one of these yields stability?

Solution: Suppose first that $a > 0$, and write

$$g(u) = au + b = a(u - \overline{u}),$$

where $\overline{u} = -\dfrac{b}{a}$ is the equilibrium point found in the previous part. Since $a > 0$, we see that $u'(t) > 0$ if $u > \overline{u}$ and $u'(t) < 0$ if $u < \overline{u}$. Thus, $u(t)$ increases for $u > \overline{u}$ and decreases for $u < \overline{u}$. To get an idea of what the concavity of the graphs of solutions is, compute

$$\begin{aligned} u''(t) &= \frac{d}{dt}(u'(t)) \\ &= \frac{d}{dt}(g(u)) \\ &= g'(u)\frac{du}{dt} \\ &= a^2(u - \overline{u}). \end{aligned}$$

Thus, we see that the graph of $u = u(t)$ is concave up for $u > \overline{u}$ and concave down is $u < \overline{u}$. Putting all the information obtained from the signs of $u'(t)$ and $u''(t)$ together, we obtain the sketch shown in Figure 1.

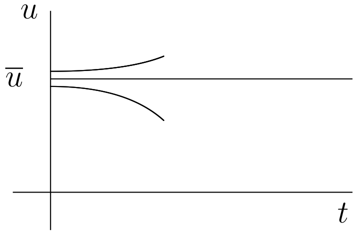

Figure 1: Possible Solutions for $a > 0$ and $b < 0$

Next, consider the case $a < 0$, so that $\overline{u} > 0$. In this case, using

$$u'(t) = a(u - \overline{u})$$

and

$$u''(t) = a^2(u - \overline{u}),$$

we see that $u(t)$ decreases for $u > \overline{u}$ and increases for $u < \overline{u}$; the graph of $u = u(t)$ is concave down for $u < \overline{u}$ and concave up for $u > \overline{u}$. A sketch of possible solutions is shown in Figure 2. The

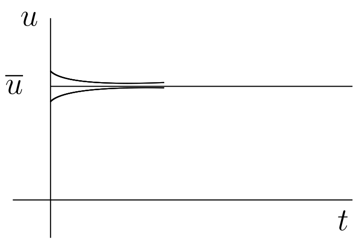

Figure 2: Possible Solutions for $a < 0$ and $b > 0$

sketch in Figure 2 suggests that $\overline{u}$ is stable for the case $a < 0$. $\square$

(c) Use separation of variables to obtain solutions to the equation.

Solution: Write the equation in the form $\dfrac{du}{dt} = a\,(u - \overline{u})\,,$ where $\overline{u} = -\dfrac{b}{a}$, and separate variables to get

$$\int \frac{1}{u - \overline{u}}\; du = \int a\; dt,$$

which yields

$$\ln|u - \overline{u}| = at + c_1,$$

for some constant c_1. Exponentiating on both sides of the previous equation, and then solving for $u = u(t)$ yields

$$u(t) = \overline{u} + Ce^{at}, \tag{1}$$

for some constant C. □

(d) Use your result from the previous part to justify your answers to part (b).

Solution: If $a < 0$, it follows from the result in equation (1) that

$$\lim_{t\to\infty} u(t) = \overline{u}.$$

Thus, $\overline{u}$ is asymptotically stable in this case.

We also get from (1) that

$$|u(t) - \overline{u}| = |C|e^{at}$$

for all $t \in \mathbf{R}$. Thus, is $a > 0$, the distance from $u(t)$ to the equilibrium point, $\overline{u}$, increases as t increases. Hence, if $a > 0$, then $\overline{u}$ is unstable. □