Review Problems for Exam 1

1. Consider the difference equation $X_{t+1} = \lambda X_t + a$, where λ and a are real parameters, given that X_0 is known.

   (a) Find a closed form solution, X_t, to the equation and discuss how the behavior of the solution as $t \to \infty$ is determined by the value of λ.

   (b) Write the difference equation in the form $X_{t+1} = f(X_t)$, for some function f.

   Give the equilibrium point(s) of the equation and use the principle of linearized stability to determine the nature of their stability.

2. Find the equilibrium point of the difference equation $X_{t+1} = X_t^2 - 6$, and determine their stability properties.

3. Suppose the growth of a population of size N_t at time t is dictated by the discrete model

$$N_{t+1} = \frac{400N_t}{(10+N_t)^2}.$$

   (a) Find the biologically reasonable fixed points for this difference equation.

   (b) Determine the stability properties of the equilibrium points found in the previous part.

   (c) If $N_0 = 5$, what happens to the population in the long run?

4. We have seen that the (continuous) logistic model $\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$, where r and K are positive parameters, has an equilibrium point at $\overline{N} = K$.

   (a) Let $g(N) = rN\left(1 - \frac{N}{K}\right)$ and give the linear approximation to $g(N)$ for N close to K:

$$g(K) + g'(K)(N - K).$$

   Observe that $g(K) = 0$ since K is an equilibrium point.

   (b) Let $u = N - K$ and consider the linear differential equation

$$\frac{du}{dt} = g'(K)u.$$

This is called the *linearization* of the equation

$$\frac{dN}{dt} = g(N)$$

around the equilibrium point $\overline{N} = K$.

Use separation of variables to solve this equation. What happens to $|u(t)|$ as $t \to \infty$, where u is any solution to the linearized equation?

(c) Use your result in the previous part to give an explanation as to why any solution to the logistic equation that begins very close to K can be approximation by $K + u(t)$, where u is a solution to the linearized equation.

(d) Suppose that $N = N(t)$ is a solution to the logistic equation that starts at N_o, where N_o is very close to K. Find an estimate of the time it takes for the distance $|N(t) - K|$ to decrease by a factor of e. This time is called the *recovery time.*

5. *[Harvesting]* The following differential equation models the growth of a population of size $N = N(t)$ that is being harvested at a rate proportional to the population density

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - EN, \tag{1}$$

where r, K and E are parameters and non–negative parameters with $r > 0$ and $K > 0$.

(a) Give an interpretation for this model. In particular, give interpretation for the term EN. The parameter E is usually called the harvesting *effort.*

(b) Calculate the equilibrium points for the equation (1), and give conditions on the parameters that yield a biologically meaningful equilibrium point. Determine the nature of the stability of that equilibrium point. Sketch possible solutions to the equation in this situation.

(c) What does the model predict if $E \geq r$?

6. *[Harvesting, continued]* Suppose that $0 < E < r$ in equation (1), and let $\overline{N}$ denote the positive equilibrium point. The quantity $Y = E\overline{N}$ is called the *harvesting yield.*

(a) Find the value of E for which the harvesting yield is the largest possible; this value of the yield is called the *maximum sustainable yield.*

(b) What is the value of the equilibrium point for which there is the maximum sustainable yield?