Solutions to Part I of Exam 2

1. Suppose that the rate at which a drug leaves the bloodstream and passes into the urine at a given time is proportional to the quantity of the drug in the blood at that time.

   (a) Write down and solve a differential equation for the quantity, $Q = Q(t)$, of the drug in the blood at time, t, in hours. State all the assumptions you make and define all the parameters that you introduce.

   Solution: By the conservation principle for a one–compartment model,
   $$\frac{\mathrm{d}Q}{\mathrm{d}t} = \text{Rate of } Q \text{ in} - \text{Rate of } Q \text{ out},$$
   where
   $$\text{Rate of } Q \text{ in} = 0$$
   and
   $$\text{Rate of } Q \text{ out} = kQ,$$
   for some constant of proportionality k. Thus, Q satisfies the differential equation
   $$\frac{\mathrm{d}Q}{\mathrm{d}t} = -kQ,$$
   which has solution
   $$Q(t) = ce^{-kt} \quad \text{for all } t \geqslant 0.$$
   for some constant c. □

   (b) Suppose that an initial dose of Q_o is injected directly into the blood, and that 20% of that initial amount is is left in the blood after 3 hours. Based on the solution you found in the previous part, write down $Q(t)$ for this situation and sketch its graph.

   Solution: If $Q(0) = Q_o$, then $c = Q_o$. Thus,
   $$Q(t) = Q_o e^{-kt} \quad \text{for all } t \geqslant 0.$$
   If $Q(3) = 0.2Q_o$, then
   $$0.2Q_o = Q_o e^{-3k},$$

from which we obtain that

$$k = -\frac{1}{3}\ln(0.2) \approx 0.54.$$

It then follows that

$$Q(t) = Q_o e^{\frac{t}{3}\ln(0.2)} \approx Q_o e^{-0.54t}.$$

□

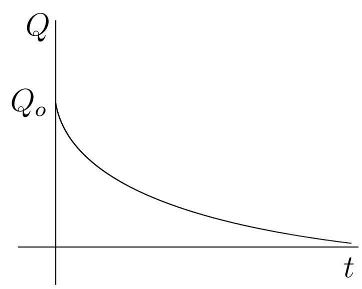

Figure 1: Sketch of graph of $Q(t)$

(c) How much of the drug is left in the patient's body after 6 hours if the patient is given 100 mg initially?

Solution: Compute

$$\begin{aligned} Q(6) &= 100e^{\frac{6}{3}\ln(0.2)} \\ &= 100e^{2\ln(0.2)} \\ &= 100\left(e^{\ln(0.2)}\right)^2 \\ &= 100(0.2)^2 \\ &= \frac{100}{25} \\ &= 4. \end{aligned}$$

Thus, there will be 4 mg of the drug left in the patient after 6 hours. □

2. Suppose a bacterial colony has N_o bacteria at time $t = 0$. Let $M(t)$ denote the number of bacteria that develop certain mutation during the time interval $[0, t]$. Assume that, for small $\Delta t > 0$,

$$M(t + \Delta t) - M(t) \cong a\ (\Delta t)\ N(t), \tag{1}$$

where a is a positive constant, and $N(t)$ is the number of bacteria in the colony at time t.

(a) Give an interpretation to what the expression in (1) is saying. In particular, provide a meaning for the constant, a, known as the *mutation rate.*

Solution: The expression in (1) postulates that the number of mutations occurring in the time interval $[t, t+\Delta t]$ is proportional to the length of the interval, Δt, and the number of cells, $N(t)$, present at time t. The constant of proportionality, a, can be interpreted as the fraction of cells that mutate in a unit of time. □

(b) Let $\mu(t) = E(M(t))$ denote the expected value of the number of mutations in the time interval $[0, t]$. It is possible to prove, using the expression in (1), that $\mu = \mu(t)$ is differentiable and satisfies the differential equation

$$\frac{d\mu}{dt} = aN(t). \tag{2}$$

Solve the differential equation in (2) assuming that $N(t)$ grows in time according to a Malthusian model with per–capita growth rate k, and that there are no mutant bacteria at time $t = 0$.

Solution: Assuming that the bacterial colony is growing according the Malthusian model

$$\begin{cases} \dfrac{dN}{dt} = kN \\ N(0) = N_o, \end{cases}$$

where $k = \dfrac{\ln 2}{T}$, T being the doubling time or the duration of a division cycle, then $N(t) = N_o e^{kt}$. Substituting this into (2) we get

$$\frac{d\mu}{dt} = aN_o e^{kt},$$

which can be integrated to yield

$$\begin{aligned} \mu(t) - \mu(0) &= \int_0^t aN_o e^{k\tau}\, \mathrm{d}\tau \\ &= \frac{a}{k} N_o(e^{kt} - 1). \end{aligned}$$

If there no mutations at time $t = 0$, $\mu(0) = 0$, and so

$$\mu(t) = \frac{a}{k}(N_o e^{kt} - N_o),$$

or

$$\mu(t) = \frac{a}{k}(N(t) - N_o).$$

Hence, the average number of mutations which occur in the interval $[0, t]$ is proportional to the population increment during that time period. The constant of proportionality is the mutation rate divided by the growth rate. $\square$