Assignment #11

Due on Monday, March 28, 2011

Read Section 2.6 on *Curves and Simple Arcs and Orientation,* pp. 45–56, in Baxandall and Liebek's text.

Read Section 2.7 on *Path Length and Length of Simple Arcs,* pp. 59–66, in Baxandall and Liebek's text.

Read Section 5.1.1 on *Arc Length* in the class Lecture Notes (pp. 64–67).

Background and Definitions

- *(Reparametrizations)* Let $\sigma\colon [a,b]\to\mathbb{R}^n$ be a differentiable, one–to–one path. Suppose also that $\sigma'(t)$, is never the zero vector. Let $h\colon [c,d]\to[a,b]$ be a differentiable, one–to–one and onto map such that $h'(t)\neq 0$ for all $t\in[c,d]$. Define
$$\gamma(t)=\sigma(h(t))\quad\text{for all }\ t\in[c,d].$$
$\gamma\colon [c,d]\to\mathbb{R}^n$ is a called a *reparametrization* of σ

- *(Arc Length Parameter)* Let I denote an open interval in $\mathbb{R}$, and $\sigma\colon I\to\mathbb{R}^n$ be a parametrization of a curve C. For fixed $a\in I$, define
$$s(t)=\int_a^t\|\sigma'(\tau)\|\,\mathrm{d}\tau\quad\text{for all }\ t\in I. \tag{1}$$
The parameter $s=s(t)$ measures the length along the curve C from the point $\sigma(a)$ to the point $\sigma(t)$.

Do the following problems

1. Show that the arc length parameter defined in (1) is differentiable on I and compute $s'(t)$ for all $t\in I$. Deduce that $s(t)$ is a strictly increasing function of t in I.

2. Let $\gamma\colon [c,d]\to\mathbb{R}^n$ be a reparametrization of $\sigma\colon [a,b]\to\mathbb{R}^n$. of σ

   (a) Show that γ is a differentiable, one–to–one path.

   (b) Compute $\gamma'(t)$ and show that it is never the zero vector.

   (c) Show that σ and γ have the same image in $\mathbb{R}^n$.

3. Let C be a curve parametrized by

$$\sigma(t) = \sigma(t) = (e^{kt}\cos t, e^{kt}\sin t), \qquad \text{for } t \in [0, 2\pi],$$

where $k \neq 0$. Compute the arc length of C.

4. A particle is following a path in three–dimensional space given by

$$\sigma(t) = (e^t, e^{-t}, 1 - t) \quad \text{for} \quad t \in \mathbb{R}.$$

At time $t_o = 1$, the particle flies off on a tangent.

(a) Where will the particle be at time $t_1 = 2$?

(b) Will the particle ever hit the xy–plane? Is so, find the location on the xy plane where the particle hits.

5. Let $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1, y \geqslant 0\}$; i.e., C is the upper unit semi–circle. C can be parametrized by

$$\sigma(\tau) = (\tau, \sqrt{1 - \tau^2}) \quad \text{for} \quad -1 \leqslant \tau \leqslant 1.$$

(a) Compute $s(t)$, the arclength along C from $(-1, 0)$ to the point $\sigma(t)$, for $-1 \leqslant t \leqslant 1$.

(b) Compute $s'(t)$ for $-1 < t < 1$ and sketch the graph of s as function of t.

(c) Show that $\cos(\pi - s(t)) = t$ for all $-1 \leqslant t \leqslant 1$, and deduce that

$$\sin(s(t)) = \sqrt{1 - t^2} \quad \text{for all} \quad -1 \leqslant t \leqslant 1.$$