## Assignment #12

**Due on Wednesday, March 30, 2011**

**Read** Section 3.6 on *The Chain Rule and the Rate of Change along a Path,* pp. 133–136, in Baxandall and Liebek's text.

**Read** Section 3.7 on *Directional Derivatives,* pp. 138–141, in Baxandall and Liebek's text.

**Read** Section 3.8 on *The Gradient and Smooth Surfaces,* pp. 142–151, in Baxandall and Liebek's text.

**Read** Section 4.6 on *Derivatives of Compositions* in the class Lecture Notes (pp. 56–60).

**Do** the following problems

1. Let $U = \{(x, y) \in \mathbb{R}^2 \mid x \neq 0\}$ and define $f\colon U \to \mathbb{R}$ by $f(x, y) = \arctan\left(\dfrac{y}{x}\right)$, for all $(x, y) \in U$.

   (a) Compute the gradient of $f$ in $U$.

   (b) Let $I$ be an open interval and $\sigma\colon I \to U$ be a differentiable path given by $\sigma(t) = (x(t), y(t))$ for $t \in I$. Define $\theta\colon I \to \mathbb{R}$ by $\theta(t) = (f \circ \sigma)(t)$ for all $t \in I$. Apply the Chain Rule to verify that $\theta' = \dfrac{-yx' + xy'}{x^2 + y^2}$.

   (c) Apply the result from part (b) to the path $\sigma\colon \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \to \mathbb{R}^2$ given by $\sigma(t) = (\cos t, \sin t)$, for $t \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

2. Suppose that the temperature in a region of space is given by a function $T\colon \mathbb{R}^3 \to \mathbb{R}$ given by

   $$T(x, y, z) = kx^2(y - z), \quad \text{for all } (x, y, z) \in \mathbb{R}^3,$$

   and some positive constant $k$.

   An insect flies in the region along a path modeled by a $C^1$ function $\sigma\colon \mathbb{R} \to \mathbb{R}^3$. Suppose that at time $t = 0$ the insect is located at $(0, 0, 0)$ and its velocity is $\sigma'(0) = \widehat{i} + \widehat{j} + 2\widehat{k}$. Compute the rate of change of temperature sensed by the insect at time $t = 0$.

3. Let $I$ be an open interval of real numbers and $U$ be an open subset of $\mathbb{R}^n$. Suppose that $\sigma\colon I \to \mathbb{R}^n$ is a differentiable path and that $f\colon U \to \mathbb{R}$ is a differentiable scalar field. Assume also that the image of $I$ under $\sigma$, $\sigma(I)$, is contained in $U$. Suppose also that the derivative of the path $\sigma$ satisfies

$$\sigma'(t) = -\nabla f(\sigma(t)) \quad \text{for all } \ t \in I.$$

   Show that if the gradient of $f$ along the path $\sigma$ is never zero, then $f$ decreases along the path as $t$ increases.

   *Suggestion:* Use the Chain Rule to compute the derivative of $f(\sigma(t))$.

4. A set $U \subseteq \mathbb{R}^n$ is said to be **path connected** iff for any vectors $x$ and $y$ in $U$, there exists a differentiable path $\sigma\colon [0,1] \to \mathbb{R}^n$ such that $\sigma(0) = x$, $\sigma(1) = y$ and $\sigma(t) \in U$ for all $t \in [0,1]$; i.e., any two elements in $U$ can be connected by a differentiable path whose image is entirely contained in $U$.

   Suppose that $U$ is an open, path connected subset of $\mathbb{R}^n$. Let $f\colon U \to \mathbb{R}$ be a differentiable scalar field such that $\nabla f(x)$ is the zero vector for all $x \in U$. Prove that $f$ must be constant.

5. Let $U$ be an open subset of $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}$ be a differentiable scalar filed defined on $U$. The function $f$ is said to be homogeneous of order $k$ if

$$f(tv) = t^k f(v),$$

   for all $v \in U$ and all positive $t \in \mathbb{R}$ such that $tv \in U$.

   (a) Show that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = xy$, for all $(x,y) \in \mathbb{R}^2$, is homogeneous of order 2.

   (b) Give examples of a scalar field which is homogenous of order 1 and of a scalar field which is homogeneous of order 0.

   (c) Prove Euler's Theorem: If $f\colon U \to \mathbb{R}$ is differentiable and homogenous of order $k$, then

$$x_1 \frac{\partial f}{\partial x_1}(x) + x_2 \frac{\partial f}{\partial x_2}(x) + \cdots + x_n \frac{\partial f}{\partial x_n}(x) = k f(x),$$

   for all $x = (x_1, x_2, \ldots, x_n) \in U$.

   *Suggestion:* Let $\sigma\colon \mathbb{R} \to \mathbb{R}^n$ be given by $\sigma(t) = tx$, for all $t \in \mathbb{R}$ and $x = (x_1, x_2, \ldots, x_n) \in U$, and apply the Chain Rule to the composition $f \circ \sigma$.