Assignment #19

Due on Wednesday, April 27, 2011

Read Section 11.3 on *Differential* 2*–Forms,* pp. 527–534, in Baxandall and Liebek's text.

Read Section 5.6 on *Calculus of Differential Forms* in the class Lecture Notes (pp. 89–91).

Read Section 5.7 on *Evaluating Differential* 2*–Forms: Double Integrals* in the class Lecture Notes (pp. 91–96).

Background and Definitions

- *(The Fundamental Theorem of Calculus in $\mathbb{R}^2$ or $\mathbb{R}^3$ for Oriented Triangles)* Let U denote an open region in $\mathbb{R}^2$ or $\mathbb{R}^3$ and T an oriented triangle contained in U. Denote the boundary of T by ∂T. If ω is any differential 1–form defined in U, the
$$\int_T d\omega = \oint_{\partial T} \omega. \tag{1}$$

- *(Green's Theorem for Oriented Triangles)* Let U denote an open region in $\mathbb{R}^2$ and T an oriented triangle contained in U. Denote the boundary of T by ∂T, and assume that it is oriented in the counterclockwise sense. For any C^1 functions, $P\colon U \to \mathbb{R}$ and $Q\colon U \to \mathbb{R}$, defined in U,
$$\iint_T \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dxdy = \oint_{\partial T} P \, dx + Q \, dy \tag{2}$$

- *(Divergence of a Vector Field in $\mathbb{R}^2$)* Given a C^1 vector field, $F(x,y) = P(x,y)\,\widehat{i} + Q(x,y)\,\widehat{j}$, defined on some open subset U of $\mathbb{R}^2$, the divergence of F is the scalar field on U given by
$$\text{div}F(x,y) = \frac{\partial P}{\partial x}(x,y) + \frac{\partial Q}{\partial y}(x,y) \quad \text{for all } (x,y) \in U. \tag{3}$$

Do the following problems

1. Evaluate the differential form $3\, dx \wedge dy$ on each of the following oriented triangles:

   (a) $[(5,2),(1,3),(3,4)]$ (b) $[(1,0,-2),(3,1,5),(-2,1,0)]$

2. Let P and Q denote smooth scalar fields defined in some open region, U, or $\mathbb{R}^2$, and define the 1–form $\omega = P\mathrm{d}y - Q\mathrm{d}x$.

   (a) Compute the differential, $\mathrm{d}\omega$, of ω.

   (b) Recall that the integral $\displaystyle\int_C \omega$, where C is a simple closed curve in U, gives the flux of the field. $F = P\,\widehat{\imath} + Q\,\widehat{\jmath}$, across the curve C.
   What does the Fundamental Theorem of Calculus in (1), where T is a positively oriented triangle in U, say about the flux of F across the boundary of T and the divergence of F as defined in (3)?

3. Verify the Fundamental Theorem of Calculus in (1) for the differential 1–form

$$\omega = yz\ dx + xz\ dy + xy\ dz,$$

   and the oriented triangle $T = [P_oP_1P_2]$, where P_o, P_1 and P_2 are any three non–collinear points in $\mathbb{R}^3$.

4. Let T denote the triangle with vertices $P_o(0,0)$, $P_1(2,0)$ and $P_2(1,1)$, where the boundary, ∂T, of T is oriented in the counterclockwise sense. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field given by

$$F(x,y) = -\frac{y}{2}\,\widehat{\imath} + \frac{x}{2}\,\widehat{\jmath}.$$

   Evaluate the line integral $\displaystyle\oint_{\partial T} F\cdot d\mathbf{r}$ by applying Green's Theorem in (2).

5. Let T and F be as in Problem 4.

   Evaluate the flux of F across ∂T, $\displaystyle\oint_{\partial T} F\cdot d\mathbf{n}$, by applying Green's Theorem in (2).