Review Problems for Exam 1

1. Compute the (shortest) distance from the point $P(4, 0, -7)$ in $\mathbb{R}^3$ to the plane given by $4x - y - 3z = 12$.

2. Compute the (shortest) distance from the point $P(4, 0, -7)$ in $\mathbb{R}^3$ to the line given by the parametric equations

$$\begin{cases} x = -1 + 4t, \\ y = -7t, \\ z = 2 - t. \end{cases}$$

3. Compute the area of the triangle whose vertices in $\mathbb{R}^3$ are the points $(1, 1, 0)$, $(2, 0, 1)$ and $(0, 3, 1)$

4. Let v and w be two vectors in $\mathbb{R}^3$, and let λ be a scalar. Show that the area of the parallelogram determined by the vectors v and $w + \lambda v$ is the same as that determined by v and w.

5. Let $\widehat{u}$ denote a unit vector in $\mathbb{R}^n$ and $P_{\widehat{u}}(v)$ denote the orthogonal projection of v along the direction of $\widehat{u}$ for any vector $v \in \mathbb{R}^n$. Use the Cauchy–Schwarz inequality to prove that the map

$$v \mapsto P_{\widehat{u}}(v) \quad \text{for all } \ v \in \mathbb{R}^n$$

is a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

6. Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$ Prove that f is continuous at $(0, 0)$.

7. Show that

$$f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is not continuous at $(0, 0)$.

8. Determine the value of L that would make the function

$$f(x,y) = \begin{cases} x\ \sin\left(\dfrac{1}{y}\right) & \text{if } \ y \neq 0; \\ \\ L & \text{otherwise }, \end{cases}$$

continuous at $(0,0)$. Is $f\colon \mathbb{R}^2 \to \mathbb{R}$ continuous on $\mathbb{R}^2$? Justify your answer.

9. Define the scalar field $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(v) = \dfrac{1}{2}\|v\|^2$ for all $v \in \mathbb{R}^n$. Show that f is differentiable on $\mathbb{R}^n$ and compute the linear map $Df(u)\colon \mathbb{R}^n \to \mathbb{R}$ for all $u \in \mathbb{R}^n$. What is the gradient of f at u for all $x \in \mathbb{R}^n$?

10. Let $g\colon\ [0,\infty) \to \mathbb{R}$ be a differentiable, real–valued function of a single variable, and let $f(x,y) = g(r)$ where $r = \sqrt{x^2+y^2}$.

    (a) Compute $\dfrac{\partial r}{\partial x}$ in terms of x and r, and $\dfrac{\partial r}{\partial y}$ in terms of y and r.

    (b) Compute ∇f in terms of $g'(r)$, r and the vector $\mathbf{r} = x\widehat{i} + y\widehat{j}$.

11. Let $f\colon U \to \mathbb{R}$ denote a scalar field defined on an open subset U of $\mathbb{R}^n$, and let $\widehat{u}$ be a unit vector in $\mathbb{R}^n$. If the limit

$$\lim_{t\to 0} \frac{f(v+t\widehat{u}) - f(v)}{t}$$

exists, we call it the *directional derivative of f at v in the direction of the unit vector $\widehat{u}$*. We denote it by $D_{\widehat{u}}f(v)$.

    (a) Show that if f is differentiable at $v \in U$, then, for any unit vector $\widehat{u}$ in $\mathbb{R}^n$, the directional derivative of f in the direction of $\widehat{u}$ at v exists, and

$$D_{\widehat{u}}f(v) = \nabla f(v)\cdot \widehat{u},$$

    where $\nabla f(v)$ is the gradient of f at v.

    (b) Suppose that $f\colon U \to \mathbb{R}$ is differentiable at $v \in U$. Prove that if $D_{\widehat{u}}f(v) = 0$ for every unit vector $\widehat{u}$ in $\mathbb{R}^n$, then $\nabla f(v)$ must be the zero vector.

    (c) Suppose that $f\colon U \to \mathbb{R}$ is differentiable at $v \in U$. Use the Cauchy–Schwarz inequality to show that the largest value of $D_{\widehat{u}}f(v)$ is $\|\nabla f(v)\|$ and it occurs when $\widehat{u}$ is in the direction of $\nabla f(v)$.