Assignment #8

Due on Wednesday, April 13, 2011

Read Section I.7 on *Autonomous Systems–Generalities,* pp. 37–46, in Hale's text.

Read Section I.8 on *Autonomous Systems–Limit Sets, Invariant Sets,* pp. 46–49, in Hale's text.

Read Chapter 4 on *Continuous Dynamical Systems,* starting on page 47, in the class lecture notes.

Background and Definitions

- *(Distance from a Point to a Set)* Given a nonempty subset, A, of $\mathbb{R}^N$, we define the distance from a point $x \in \mathbb{R}^N$ to the set A as follows

$$\text{dist}(x, A) = \inf_{y \in A} \|y - x\|. \tag{1}$$

- *(Distance Between Two Sets)* Given two nonempty subsets, A and B, of $\mathbb{R}^N$, the distance from A to B, denoted by $\text{dist}(A, B)$, is defined by

$$\text{dist}(A, B) = \inf_{x \in B} \text{dist}(x, A). \tag{2}$$

- *(Proper Maps)* Let U be an open subset of $\mathbb{R}^N$. A continuous function $F: U \to \mathbb{R}^k$ is said to be a proper map if the inverse image of every compact subset in $\mathbb{R}^k$ is compact.

1. Let A be a nonempty subset of $\mathbb{R}^N$. Prove that

$$|\text{dist}(x, A) - \text{dist}(y, A)| \leqslant \|x - y\|, \quad \text{for all } x, y \in \mathbb{R}^N.$$

Deduce therefore that the function $f: \mathbb{R}^N \to \mathbb{R}$ defined by $f(x) = \text{dist}(x, A)$ is Lipschitz continuous.

Suggestion: Apply the triangle inequality to obtain

$$\text{dist}(x, A) \leqslant \|z - y\| + \|y - x\|, \quad \text{for all } z \in A.$$

2. Let $\theta\colon \mathbb{R}\times U \to U$ be a dynamical system in an open subset, U, of $\mathbb{R}^N$. For $p\in U$, assume that γ_p^+ is contained in a compact subset of U. Prove that

$$\lim_{t\to\infty} \text{dist}(\theta(t,p),\omega(\gamma_p)) = 0$$

Suggestion: Argue by contradiction; that is, start out assuming that there exists $\varepsilon_o > 0$ and a sequence of positive real numbers, (t_m), such that $t_m \to \infty$ as $m\to\infty$, and

$$\text{dist}(\theta(t_m,p),\omega(\gamma_p)) \geqslant \varepsilon_o, \qquad \text{for all } m = 1,2,3,\ldots$$

Observe that the sequence $(\theta(t_m,p))$ lies in a compact subset of U.

3. Let $V(x) = \|x\|^2$ for all $x\in\mathbb{R}^N$. Prove that V defines a proper map.

Suggestion: Compute $V^{-1}([a,b])$ for every closed and bounded interval, $[a,b]$.

4. Let U be an open subset of $\mathbb{R}^N$ and let $V\colon U\to\mathbb{R}$ be a C^2 function. Put $F(x) = -\nabla V(x)$ for all $x\in U$. Assume that V has a (strict) global minimum at $\overline{x}\in U$; that is,

$$V(\overline{x}) < V(y), \qquad \text{for all } y\in U\backslash\{\overline{x}\}.$$

Prove that if V is a proper map, and F has no critical points in $U\backslash\{\overline{x}\}$, then

$$\omega(\gamma_p) = \{\overline{x}\}, \qquad \text{for all } p\in U.$$

We say that $\overline{x}$ is a global attractor.

5. Consider the two–dimensional system

$$\begin{cases} \dfrac{dx}{dt} = y + \dfrac{x}{\sqrt{x^2+y^2}}(x^2+y^2-4); \\[2ex] \dfrac{dy}{dt} = -x + \dfrac{y}{\sqrt{x^2+y^2}}(x^2+y^2-4). \end{cases} \tag{3}$$

By expressing the system (3) in terms of the variables r and θ, where $x = r\cos\theta$ and $y = r\sin\theta$, obtain a picture of the phase portrait of the system. Determine equilibrium solutions and periodic solutions, if any, of the system. Discuss the limiting behavior of the system.