Assignment #9

Due on Monday, April 25, 2011

Read Section II.1 on *Two–Dimensional Systems,* pp. 51–63, in Hale's text.

Read Chapter X on *The Direct Method of Liapunov,* pp. 311–319, in Hale's text.

Read Chapter 4 on *Continuous Dynamical Systems,* starting on page 47, in the class lecture notes.

Background and Definitions

- In the problems in this problem set, a dot on top of a variable will denote the derivative with respect to t of that variable; for instance $\dot{x} = \dfrac{dx}{dt}$.
- If V is a real valued C^1 function defined on an open region U in $|Rtwo$, we define $\dot{V}(x,y)$ to by

$$\dot{V}(x,y) = \frac{d}{dt}(V(\varphi_t(x,y)) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} = \frac{\partial V}{\partial x}f(x,y) + \frac{\partial V}{\partial y}g(x,y).$$

  That is, $\dot{V}(x,y)$ is the rate of change of V along the orbit of system of the 2–dimensional system

$$\begin{cases} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y), \end{cases} \tag{1}$$

  going through (x,y).

1. Let A and Q be 2×2 matrices, and assume that Q is invertible. Suppose that $(x(t),y(t))$ is a solution to the system $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = A\begin{pmatrix}x\\ y\end{pmatrix}$. Show that by making the change of variables $\begin{pmatrix}u\\ v\end{pmatrix} = Q^{-1}\begin{pmatrix}x\\ y\end{pmatrix}$, we obtain a solution to the system

$$\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = Q^{-1}AQ\begin{pmatrix}u\\ v\end{pmatrix}.$$

2. Consider the linear system $\begin{cases} \dot{x} &= -y \\ \dot{y} &= 2x+3y \end{cases}$

   (a) Write the system in the matrix form $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = A\begin{pmatrix}x\\ y\end{pmatrix}$.

(b) Let $Q = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$, and set $Q^{-1}AQ = J$. Give the general solution of the system $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = J \begin{pmatrix} u \\ v \end{pmatrix}$, and sketch the phase portrait in the uv–plane.

(c) Give the general solution of the system $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$. and sketch the phase portrait.

(d) Determine the nature of the stability of the equilibrium point $(0, 0)$.

3. Let $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ denote a solution of the two–dimensional linear system $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$, and suppose that $\lim_{t\to\infty} \sqrt{(x(t))^2 + (y(t))^2} = 0$. Use the continuity of the linear transformation $\begin{pmatrix} u \\ v \end{pmatrix} = Q^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$, where Q is an invertible 2×2 matrix, to show that the solution $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = Q^{-1} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ of the system

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = J \begin{pmatrix} u \\ v \end{pmatrix}, \quad \text{where } J = Q^{-1}AQ,$$

also satisfies the property $\lim_{t\to\infty} \sqrt{(u(t))^2 + (v(t))^2} = 0$.

Hence, prove that if $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$ has an asymptotically stable equilibrium point, $(0, 0)$, then so does the system $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = J \begin{pmatrix} u \\ v \end{pmatrix}$, and vice versa.

4. Assume that the functions f and g in the system (1) are C^1 functions defined on all of $\mathbb{R}^2$. Let $V : \mathbb{R}^2 \to \mathbb{R}$ denote a C^1 function satisfying

$$V(x, y) \to \infty \quad \text{as} \quad ||(x, y)|| \to \infty,$$

and $\dot{V}(x, y) \leq 0$ for all $(x, y) \in \mathbb{R}^2$.

(a) Show that the set $\{\theta(t, x, y) \mid t \geqslant 0\}$ is bounded for any $(x, y) \in \mathbb{R}^2$.

(b) Show that $\dot{V}(\overline{x}, \overline{y}) = 0$ for all $(\overline{x}, \overline{y}) \in \omega(\gamma_{(p,q)})$ and any $(p, q) \in \mathbb{R}^2$.

5. Let $V(x, y) = ax^2 + 2bxy + cy^2$, where a, b and c are real numbers. Show that if $a > 0$ and $ac - b^2 > 0$, then V is positive definite in $\mathbb{R}^2$.