Exam 1 (Part I)

Wednesday, March 2, 2011

Name: ______________________________

This is the in–class part of Exam 1. It is a closed–notes and closed–book exam. Use your own paper and/or the paper provided for you. Please, provide complete solutions. Write you name on this page and staple it to your solutions. You have 75 minutes to work on the following 3 problems. Relax.

1. Let I denote and open interval of real numbers and U denote an open subset of $\mathbb{R}^N$. Suppose that $F\colon I \times U \to \mathbb{R}^N$ is continuous. Let $(t_o, p_o) \in I \times U$.

   (a) State the local existence and uniqueness theorem for the IVP

   $$\begin{cases} \dfrac{dx}{dt} = F(t,x); \\ x(t_o) = p_o. \end{cases} \tag{1}$$

   (b) Let $I = \mathbb{R}$ and $U = \mathbb{R}$ and put $f(t,x) = 3tx^{1/3}$ for all $(t,x) \in I \times U$. Show that the IVP

   $$\begin{cases} \dfrac{dx}{dt} = f(t,x); \\ x(0) = 0, \end{cases}$$

   has more than one solution. Explain why this result does not contradict the local existence and uniqueness theorem stated in part (a).

2. Let U be an open subset of $\mathbb{R}^N$ and $F\colon U \to \mathbb{R}^N$ be a C^1 vector field. For $p \in U$, let $u_p\colon J_p \to U$ be the unique solution to the IVP

   $$\begin{cases} \dfrac{dx}{dt} = F(x); \\ x(0) = p, \end{cases} \tag{2}$$

   defined on a maximal interval of existence, J_p.

   Suppose that there exist t_1 and t_2 in J_p such that $t_1 \neq t_2$ and

   $$u_p(t_1) = u_p(t_2).$$

   Prove that $u_p(t) = u_p(t + t_2 - t_1)$, for all $t \in J_p$.

3. Let U denote an open subset of $\mathbb{R}^N$ and $F\colon U \to \mathbb{R}^N$ be a C^1 vector field defined on U.

   (a) Define the flow domain, $\mathcal{D}$, of the field F.

   (b) Define the flow map, $\theta\colon \mathcal{D} \to U$, of the field F.

   (c) Let $t > 0$ and suppose that $(t, p) \in \mathcal{D}$. It was proved in class that for any $T > t$ such that $(T, p) \in \mathcal{D}$, there exist positive constants, K and $r = r(T)$, such that $\overline{B}_r(p) \subset U$ and $\|q - p\| \leqslant r$ implies that

   $$\|\theta(t, q) - \theta(t, p)\| \leqslant \|q - p\| e^{Kt}, \qquad \text{for all } t \in [0, T]. \tag{3}$$

   Use the estimate in (3) to prove that the flow map, $\theta\colon \mathcal{D} \to U$, is continuous on $\mathcal{D}$.

   (d) When does the flow map, $\theta(t, p)$, define a continuous dynamical system?