Review Problems for Exam 1

1. Show that the IVP $\begin{cases} \dfrac{dx}{dt} = 3x^{2/3}; \\ x(0) = 0, \end{cases}$ has more than one solution. Explain why this result does not contradict the local existence and uniqueness theorem proved in class.

2. Find the maximal interval of existence, J_2, of the IVP $\begin{cases} \dfrac{dx}{dt} = x^3; \\ x(0) = 2. \end{cases}$ If one of the endpoints of the interval J_2 is finite, study the behavior of the solution $u = u(t)$ as t approaches that endpoint from within J_2. Explain your result in light of the Escape in Finite Time Theorem proved in class.

3. Show that the IVP $\begin{cases} \dfrac{dx}{dt} = \dfrac{1}{2x}; \\ x(1) = 1, \end{cases}$ has maximal interval of existence $J_1 = (0, \infty)$. Verify that the solution $u = u(t)$ is defined and continuous on $[0, \infty)$; however, u is not differentiable at 0.

4. Let J denote an open interval of real numbers and suppose that f, g and h are continuous functions defined on J. Prove that the IVP

$$\begin{cases} \dfrac{d^2z}{dt^2} + g(t)\dfrac{dz}{dt} + h(t)z = f(t); \\ z(t_o) = p \\ z'(t_o) = q, \end{cases}$$

   where $t_o \in J$ and p and q are real numbers, has a unique solutions defined on J. Prove also that solution $u(t, p, q)$ depends continuously on (p, q).

5. Find the dynamical, $\theta(t, p, q)$, for $(t, p, q) \in \mathbb{R}^3$, corresponding to the differential equations $\begin{cases} \dfrac{dx}{dt} = -x + y; \\ \dfrac{dy}{dt} = 2y. \end{cases}$ Compute the orbit of the point $(1, 3)$.

6. Let $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ be given by $F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ -y + x^2 \\ z + x^2 \end{pmatrix}$, for all $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.

   (a) Determine the flow domain, $\mathcal{D}$, of the vector field F, and compute the flow map , $\theta(t,p,q,r)$, for $(t,p,q,r) \in \mathcal{D}$.

   (b) Let $V(t,p,q,r) = D\theta_t(t,p,q,r)$, where $\theta_t\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the map given by

   $$\theta_t(p,q,r) = \theta(t,p,q,r), \qquad \text{for all } (p,q,r) \in \mathbb{R}^3,$$

   and verify that the map $t \mapsto V(t,p,q,r)$ solves the matrix IVP

   $$\begin{cases} \dfrac{dY}{dt} &= A(t,p,q,r)Y; \\ Y(0) &= I, \end{cases}$$

   where $A(t,p,q,r) = DF(\theta(t,p,q,r))$.

7. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ 2y + x^2 \end{pmatrix}$, for all $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

   (a) Determine the flow domain, $\mathcal{D}$, of the vector field F, and compute the flow map , $\theta(t,p,q)$, for $(t,p,q) \in \mathcal{D}$.

   (b) Let $A = \{(p,q) \in \mathbb{R}^2 \mid q = -p^2/4\}$. Prove that A is invariant under the flow θ_t.

8. For $\omega > 0$, the system

   $$\begin{cases} \dfrac{dx}{dt} &= y; \\ \dfrac{dy}{dt} &= -\omega^2 x, \end{cases} \qquad (1)$$

   models a harmonic oscillator.

   For a point (p,q) in the xy–plane, let $\gamma_{(p,q)}$ denote the orbit of (p,q) under the flow of the system in (1).

   (a) Let $(x(t),y(t)) \in \gamma_{(p,q)}$ for all $t \in \mathbb{R}$. Verify that $\dfrac{d}{dt}[\omega^2 x^2 + y^2] = 0$, for all $t \in \mathbb{R}$.

(b) Deduce from part (a) that $\gamma_{(p,q)}$ is the graph of $\omega^2 x^2 + y^2 = C$, for some constant C. What is C in terms of p and q?

(c) Sketch the graph of some typical orbits of the system in (1).