

Assignment #18

Due on Wednesday, April 11, 2012

Read Section 5.4 on *The Poisson Distribution* in DeGroot and Schervish.

Read Section 5.6 on *The Normal Distribution* in DeGroot and Schervish.

Background and Definitions

Definition (Convergence in Distribution). Let (X_n) be a sequence of random variables with cumulative distribution functions F_{X_n} , for $n = 1, 2, 3, \dots$, and Y be a random variable with cdf F_Y . We say that the sequence (X_n) converges to Y in distribution, if

$$\lim_{n \rightarrow \infty} F_{X_n}(x) = F_Y(x)$$

for all x where F_Y is continuous. The distribution of Y is usually called the **limiting distribution** of the sequence (X_n) .

Theorem (mgf Convergence Theorem). Let (X_n) be a sequence of random variables with moment generating functions $\psi_{X_n}(t)$, for $|t| < h$, $n = 1, 2, 3, \dots$, and some positive number h . Suppose Y has mgf $\psi_Y(t)$ which exists for $|t| < h$. Then, if

$$\lim_{n \rightarrow \infty} \psi_{X_n}(t) = \psi_Y(t), \quad \text{for } |t| < h,$$

it follows that

$$\lim_{n \rightarrow \infty} F_{X_n}(x) = F_Y(x)$$

for all x where F_Y is continuous.

Do the following problems

1. Let a denote a real number and X_a be a discrete random variable with pmf

$$p_{X_a}(x) = \begin{cases} 1 & \text{if } x = a; \\ 0 & \text{elsewhere.} \end{cases}$$

- (a) Compute the cdf for X_a and sketch its graph.
- (b) Compute the mgf for X_a and determine $E(X_a)$ and $\text{var}(X_a)$.

2. Let (X_k) denote a sequence of independent identically distributed random variables such that $X_k \sim \text{Normal}(\mu, \sigma^2)$ for every $k = 1, 2, \dots$, and for some $\mu \in \mathbb{R}$ and $\sigma > 0$. For each $n \geq 1$, define

$$\bar{X}_n = \frac{X_1 + X_2 + \dots + X_n}{n}.$$

- (a) Determine the mgf, $\psi_{\bar{X}_n}(t)$, for \bar{X}_n , and compute $\lim_{n \rightarrow \infty} \psi_{\bar{X}_n}(t)$.
- (b) Find the limiting distribution of \bar{X}_n as $n \rightarrow \infty$. (*Hint:* Compare your answer in part (a) to your answer in part (b) of problem 1.)
3. Let (X_k) and \bar{X}_n be defined as in the previous problem. Define $Z_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$ for all $n \geq 1$.
- (a) Determine the mgf, $\psi_{Z_n}(t)$, for Z_n , and compute $\lim_{n \rightarrow \infty} \psi_{Z_n}(t)$.
- (b) Find the limiting distribution of Z_n as $n \rightarrow \infty$.

4. Let (Y_n) be a sequence of discrete random variables having pmfs

$$p_{Y_n}(y) = \begin{cases} 1 & \text{if } y = n, \\ 0 & \text{elsewhere.} \end{cases}$$

Compute the mgf of Y_n for each $n = 1, 2, 3, \dots$

Does $\lim_{n \rightarrow \infty} \psi_{Y_n}(t)$ exist for any t in an open interval around 0?

Does the sequence (Y_n) have a limiting distribution? Justify your answer.

5. Let $q = 0.95$ denote the probability that a person, in certain age group, lives at least 5 years.
- (a) If we observe 60 people from that group and assume independence, what is the probability that at least 56 of them live 5 years or more?
- (b) Find an approximation to the result of part (a) using the Poisson distribution.