

Assignment #22

Due on Monday, April 23, 2012

Read Section 8.2 on *The Chi-Square Distribution* in DeGroot and Schervish.

Read Section 8.3 on the *Joint Distribution of the Sample Mean and the Variance* in DeGroot and Schervish.

Do the following problems

1. Let (X_k) denote a sequence of independent, identically distributed $\text{Normal}(\mu, \sigma^2)$ random variables. In this problem we consider two ways of estimating the variance σ^2 based on random samples of size n , X_1, X_2, \dots, X_n .

(a) We can estimate σ^2 by using the estimator $\hat{\sigma}_n^2 = \frac{1}{n} \sum_{k=1}^n (X_k - \bar{X}_n)^2$.

The estimator $\hat{\sigma}_n^2$ is called the maximum likelihood estimator for σ^2 .

Compute $E(\hat{\sigma}_n^2)$. Is $\hat{\sigma}_n^2$ an unbiased estimator for σ^2 ?

(b) The sample variance, S_n^2 , is defined by $S_n^2 = \frac{1}{n-1} \sum_{k=1}^n (X_k - \bar{X}_n)^2$.

Compute $E(S_n^2)$. Is S_n^2 an unbiased estimator for σ^2 ?

2. **The Gamma Function.** The gamma function, $\Gamma(x)$, plays a very important role in the definitions a several probability distributions which are very useful in applications. It is defined as follows:

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \quad \text{for all } x > 0. \quad (1)$$

Note: $\Gamma(x)$ can also be defined for negative values of x which are not integers; it is not defined at $x = 0$. In this course, we will only consider $\Gamma(x)$ for $x > 0$.

Derive the following identities:

(a) $\Gamma(1) = 1$.

(b) $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$.

(c) $\Gamma(n+1) = n!$ for all non-negative integers n .

3. Let $\Gamma: (0, \infty) \rightarrow \mathbb{R}$ be as defined in (1).

(a) Compute $\Gamma(1/2)$.

Hint: The change of variable $t = z^2/2$ might come in handy. Recall that if $Z \sim \text{Normal}(0, 1)$, then its pdf is given by

$$f_z(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}} \quad \text{for all } z \in \mathbb{R}.$$

(b) Compute $\Gamma(3/2)$.

4. Use the results of Problems 2 and 3 to derive the identity:

$$\Gamma\left(\frac{k}{2}\right) = \frac{\Gamma(k)\sqrt{\pi}}{2^{k-1} \Gamma\left(\frac{k+1}{2}\right)}$$

for every positive, odd integer k .

Suggestion: Proceed by induction on k .

5. Let α and β denote positive real numbers and define $f(x) = Cx^{\alpha-1}e^{-x/\beta}$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$, where C denotes a positive real number.

(a) Find the value of C so that f is the pdf for some distribution.

(b) For the value of C found in part (a), let f denote the pdf of a random variable X . Compute the mgf of X .

Hint: The pdf found in part (a) is related to the Gamma function.