Solutions to Review Problems for Exam 1

1. There are 5 red chips and 3 blue chips in a bowl. The red chips are numbered $1, 2, 3, 4, 5$ respectively, and the blue chips are numbered $1, 2, 3$ respectively. If two chips are to be drawn at random and without replacement, find the probability that these chips are have either the same number or the same color.

   ***Solution**:* Let R denote the event that the two chips are red. Then the assumption that the chips are drawn at random and without replacement implies that

$$\Pr(R) = \frac{\binom{5}{2}}{\binom{8}{2}} = \frac{5}{14}.$$

   Similarly, if B denotes the event that both chips are blue, then

$$\Pr(B) = \frac{\binom{3}{2}}{\binom{8}{2}} = \frac{3}{28}.$$

   It then follows that the probability that both chips are of the same color is

$$\Pr(R \cup B) = \Pr(R) + \Pr(B) = \frac{13}{28},$$

   since R and B are mutually exclusive.
   Let N denote the event that both chips show the same number. Then,

$$\Pr(N) = \frac{3}{\binom{8}{2}} = \frac{3}{28}.$$

   Finally, since $R \cup B$ and N are mutually exclusive, then the probability that the chips are have either the same number or the same color is

$$\Pr(R \cup B \cup N) = \Pr(R \cup B) + \Pr(N) = \frac{13}{28} + \frac{3}{28} = \frac{16}{28} = \frac{2}{7}.$$

   □

2. A person has purchased 10 of 1,000 tickets sold in a certain raffle. to determine the five prize winners, 5 tickets are drawn at random and without replacement. Compute the probability that this person will win at least one prize.

   ***Solution**:* Let N denote the event that the person will not win any prize. Then

$$\Pr(N) = \frac{\binom{995}{10}}{\binom{1000}{10}}; \tag{1}$$

   that is, the probability of purchasing 10 non–winning tickets.
   It follows from (1) that

$$\begin{aligned} \Pr(N) &= \frac{(990)(989)(988)(987)(986)}{(1000)(999)(998)(997)(996)} \\ &= \frac{435841667261}{458349513900} \qquad (2) \\ &\approx 0.9509. \end{aligned}$$

   Thus, using the result in (2), the probability of the person winning at least one of the prizes is

$$\begin{aligned} \Pr(N^c) &= 1 - \Pr(N) \\ &\approx 1 - 0.9509 \\ &= 0.0491, \end{aligned}$$

   or about 4.91%. □

3. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and let E_1, E_2 and E_3 be mutually disjoint events in $\mathcal{B}$. Find $\Pr[(E_1 \cup E_2) \cap E_3]$ and $\Pr(E_1^c \cup E_2^c)$.

   ***Solution**:* Since E_1, E_2 and E_3 are mutually disjoint events, it follows that $(E_1 \cup E_2) \cap E_3 = \emptyset$; so that

$$\Pr[(E_1 \cup E_2) \cap E_3] = 0.$$

Next, use De Morgan's law to compute

$$\begin{aligned}
\Pr(E_1^c \cup E_2^c) &= \Pr([E_1 \cap E_2]^c) \\
&= \Pr(\emptyset^c) \\
&= \Pr(\mathcal{C}) \\
&= 1.
\end{aligned}$$

□

4. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and let A and B events in $\mathcal{B}$. Show that
$$\Pr(A \cap B) \leq \Pr(A) \leq \Pr(A \cup B) \leq \Pr(A) + \Pr(B). \tag{3}$$

***Solution**:* Since $A \cap B \subseteq A$, it follows that
$$\Pr(A \cap B) \leqslant \Pr(A). \tag{4}$$
Similarly, since $A \subseteq A \cup B$, we get that
$$\Pr(A) \leqslant \Pr(A \cup B). \tag{5}$$
Next, use the identity
$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B),$$
and fact that that
$$\Pr(A \cap B) \geqslant 0,$$
to obtain that
$$\Pr(A \cup B) \leqslant \Pr(A) + \Pr(B). \tag{6}$$
Finally, combine (4), (5) and (6) to obtain (3). □

5. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and let E_1, E_2 and E_3 be mutually independent events in $\mathcal{B}$ with probabilities $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$, respectively. Compute the exact value of $\Pr(E_1 \cup E_2 \cup E_3)$.

***Solution**:* First, use De Morgan's law to compute

$$\Pr[(E_1 \cup E_2 \cup E_3)^c] \;=\; \Pr(E_1^c \cap E_2^c \cap E_3^c) \qquad (7)$$

Then, since E_1, E_2 and E_3 are mutually independent events, it follows from (7) that

$$\Pr[(E_1 \cup E_2 \cup E_3)^c] \;=\; \Pr(E_1^c) \cdot \Pr(E_2^c) \cdot \Pr(E_3^c),$$

so that

$$\begin{aligned} \Pr[(E_1 \cup E_2 \cup E_3)^c] &= (1 - \Pr(E_1))(1 - \Pr(E_2))(1 - \Pr(E_3)) \\ &= \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) \\ &= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4}, \end{aligned}$$

so that

$$\Pr[(E_1 \cup E_2 \cup E_3)^c] \;=\; \frac{1}{4}. \qquad (8)$$

It then follows from (8) that

$$\Pr(E_1 \cup E_2 \cup E_3) = 1 - \Pr[(E_1 \cup E_2 \cup E_3)^c] = \frac{3}{4}.$$

□

6. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and let E_1, E_2 and E_3 be mutually independent events in $\mathcal{B}$ with $\Pr(E_1) = \Pr(E_2) = \Pr(E_3) = \dfrac{1}{4}$. Compute $\Pr[(E_1^c \cap E_2^c) \cup E_3]$.

***Solution**:* First, use De Morgan's law to compute

$$\Pr[((E_1^c \cap E_2^c) \cup E_3)^c] \;=\; \Pr[(E_1^c \cap E_2^c)^c \cap E_3^c] \qquad (9)$$

Next, use the assumption that E_1, E_2 and E_3 are mutually independent events to obtain from (9) that

$$\Pr[((E_1^c \cap E_2^c) \cup E_3)^c] \;=\; \Pr[(E_1^c \cap E_2^c)^c] \cdot \Pr[E_3^c], \qquad (10)$$

where

$$\Pr[E_3^c] = 1 - \Pr(E_3) = \frac{3}{4}, \tag{11}$$

and

$$\begin{aligned} \Pr[(E_1^c \cap E_2^c)^c] &= 1 - \Pr[E_1^c \cap E_2^c] \\ &= 1 - \Pr[E_1^c] \cdot \Pr[E_2^c], \end{aligned} \tag{12}$$

by the independence of E_1 and E_2.

It follows from the calculations in (13) that

$$\begin{aligned} \Pr[(E_1^c \cap E_2^c)^c] &= 1 - (1 - \Pr[E_1])(1 - \Pr[E_2]) \\ &= 1 - \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{4}\right) \\ &= 1 - \frac{3}{4} \cdot \frac{3}{4} \\ &= \frac{7}{16} \end{aligned} \tag{13}$$

Substitute (11) and the result of the calculations in (13) into (10) to obtain

$$\Pr[((E_1^c \cap E_2^c) \cup E_3)^c] = \frac{7}{16} \cdot \frac{3}{4} = \frac{21}{64}. \tag{14}$$

Finally, use the result in (14) to compute

$$\begin{aligned} \Pr[(E_1^c \cap E_2^c) \cup E_3^c] &= 1 - \Pr[((E_1^c \cap E_2^c) \cup E_3)^c] \\ &= 1 - \frac{21}{64} \\ &= \frac{43}{64}. \end{aligned}$$

□

7. A bowl contains 10 chips of the same size and shape. One and only one of these chips is red. Draw chips from the bowl at random, one at a time and without replacement, until the red chip is drawn. Let X denote the number of draws needed to get the red chip.

(a) Find the pmf of X.

Solution: Compute

$$
\begin{aligned}
\Pr(X=1) &= \frac{1}{10} \\
\Pr(X=2) &= \frac{9}{10}\cdot\frac{1}{9} = \frac{1}{10} \\
\Pr(X=3) &= \frac{9}{10}\cdot\frac{8}{9}\cdot\frac{1}{8} = \frac{1}{10} \\
&\vdots \\
\Pr(X=10) &= \frac{1}{10}
\end{aligned}
$$

Thus,

$$
p_X(k) = \begin{cases} \dfrac{1}{10} & \text{for } k = 1, 2, \ldots, 10; \\ \\ 0 & \text{elsewhere.} \end{cases} \tag{15}
$$

□

(b) Compute $\Pr(X \leq 4)$.

Solution: Use (15) to compute

$$
\Pr(X \leqslant 4) = \sum_{k=1}^{4} p_X(k) = \frac{4}{10} = \frac{2}{5}.
$$

□

8. Let X have pmf given by $p_X(x) = \dfrac{1}{3}$ for $x = 1, 2, 3$ and $p(x) = 0$ elsewhere. Give the pmf of $Y = 2X + 1$.

Solution: Note that the possible values for Y are 3, 5 and 7
Compute

$$
\Pr(Y=3) = \Pr(2X+1=3) = \Pr(X=1) = \frac{1}{3}.
$$

Similarly, we get that

$$\Pr(Y=5) = \Pr(X=2) = \frac{1}{3},$$

and

$$\Pr(Y=7) = \Pr(X=3) = \frac{1}{3}.$$

Thus,

$$p_Y(k) = \begin{cases} \dfrac{1}{3} & \text{for } k = 3, 5, 7; \\ \\ 0 & \text{elsewhere.} \end{cases}$$

□

9. Let X have pmf given by $p_X(x) = \left(\frac{1}{2}\right)^x$ for $x = 1, 2, 3, \ldots$ and $p_X(x) = 0$ elsewhere. Give the pmf of $Y = X^3$.

***Solution**:* Compute, for $y = k^3$, for $k = 1, 2, 3, \ldots$,

$$\Pr(Y = y) = \Pr(X^3 = k^3) = \Pr(X = k) = \left(\frac{1}{2}\right)^k,$$

so that

$$\Pr(Y=y) = \left(\frac{1}{2}\right)^{y^{1/3}}, \qquad \text{for } y = k^3, \quad \text{for some } k = 1, 2, 3, \ldots$$

Thus,

$$p_Y(y) = \begin{cases} \left(\dfrac{1}{2}\right)^{y^{1/3}}, & \text{for } y = k^3, \quad \text{for some } k = 1, 2, 3, \ldots; \\ \\ 0 & \text{elsewhere.} \end{cases}$$

□

10. Let $f_X(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } 1 < x < \infty; \\ \\ 0 & \text{if } x \leq 1, \end{cases}$ be the pdf of a random variable X. If E_1 denote the interval $(1, 2)$ and E_2 the interval $(4, 5)$, compute $\Pr(E_1)$, $\Pr(E_2)$, $\Pr(E_1 \cup E_2)$ and $\Pr(E_1 \cap E_2)$.

Solution: Compute

$$\Pr(E_1) = \int_1^2 \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_1^2 = \frac{1}{2},$$

$$\Pr(E_2) = \int_4^5 \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_4^5 = \frac{1}{20},$$

$$\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2) = \frac{11}{20},$$

since E_1 and E_2 are mutually exclusive, and

$$\Pr(E_1 \cap E_2) = 0,$$

since E_1 and E_2 are mutually exclusive. □

11. A *mode* of a distribution of a random variable X is a value of x that maximizes the pdf or the pmf. If there is only one such value, it is called *the mode of the distribution.* Find the mode for each of the following distributions:

(a) $p(x) = \left(\dfrac{1}{2}\right)^x$ for $x = 1, 2, 3, \ldots$, and $p(x) = 0$ elsewhere.

Solution: Note that $p(x)$ is decreasing; so, $p(x)$ is maximized when $x = 1$. Thus, 1 is the mode of the distribution of X. □

(b) $f(x) = \begin{cases} 12x^2(1 - x), & \text{if } 0 < x < 1; \\ \\ 0 & \text{elsewhere.} \end{cases}$

Solution: Maximize the function f over $[0, 1]$.
Compute

$$f'(x) = 24x(1 - x) - 12x^2 = 12x(2 - 3x),$$

so that f has a critical points at $x = 0$ and $x = \dfrac{2}{3}$.
Since $f(0) = f(1) = 0$ and $f(2/3) > 0$, it follows that f takes on its maximum value on $[0,1]$ at $x = \dfrac{2}{3}$. Thus, the mode of the distribution of X is $x = \dfrac{2}{3}$. □

12. Let X have pdf $f_X(x) = \begin{cases} 2x, & \text{if } 0 < x < 1; \\ 0, & \text{elsewhere.} \end{cases}$

Compute the probability that X is at least 3/4, given that X is at least 1/2.

***Solution**:* We are asked to compute

$$\Pr(X \geqslant 3/4 \mid X \geqslant 1/2) = \frac{\Pr[(X \geqslant 3/4) \cap (X \geqslant 1/2)]}{\Pr(X \geqslant 1/2)}, \qquad (16)$$

where

$$\begin{aligned} \Pr(X \geqslant 1/2) &= \int_{1/2}^{1} 2x\ dx \\ &= x^2 \Big|_{1/2}^{1} \\ &= 1 - \frac{1}{4}, \end{aligned}$$

so that

$$\Pr(X \geqslant 1/2) = \frac{3}{4}; \qquad (17)$$

and

$$\begin{aligned} \Pr[(X \geqslant 3/4) \cap (X \geqslant 1/2)] &= \Pr(X \geqslant 3/4) \\ &= \int_{3/4}^{1} 2x\ dx \\ &= x^2 \Big|_{3/4}^{1} \\ &= 1 - \frac{9}{16}, \end{aligned}$$

so that

$$\Pr[(X \geqslant 3/4) \cap (X \geqslant 1/2)] = \frac{7}{16}. \tag{18}$$

Substituting (18) and (17) into (16) then yields

$$\Pr(X \geqslant 3/4 \mid X \geqslant 1/2) = \frac{\frac{7}{16}}{\frac{3}{4}} = \frac{7}{12}.$$

□

13. Divide a segment at random into two parts. Find the probability that the largest segment is at least three times the shorter.

***Solution**:* Assume the segment is the interval $(0, 1)$ and let $X \sim \text{Uniform}(0, 1)$. Then X models a random point in $(0, 1)$. We have two possibilities: Either $X \leqslant 1 - X$ or $X > 1 - X$; or, equivalently, $X \leqslant \frac{1}{2}$ or $X > \frac{1}{2}$.
Define the events

$$E_1 = \left(X \leqslant \frac{1}{2}\right) \quad \text{and } E_2 = \left(X > \frac{1}{2}\right).$$

Observe that $\Pr(E_1) = \frac{1}{2}$ and $\Pr(E_2) = \frac{1}{2}$.
The probability that the largest segment is at least three times the shorter is given by

$$\Pr(E_1)\Pr(1 - X > 3X \mid E_1) + \Pr(E_2)\Pr(X > 3(1 - X) \mid E_2),$$

by the Law of Total Probability, where

$$\Pr(1 - X > 3X \mid E_1) = \frac{\Pr[(X < 1/4) \cap E_1]}{\Pr(E_1)} = \frac{1/4}{1/2} = \frac{1}{2}.$$

Similarly,

$$\Pr(X > 3(1 - X) \mid E_2) = \frac{\Pr[(X > 3/4) \cap E_1]}{\Pr(E_2)} = \frac{1/4}{1/2} = \frac{1}{2}.$$

Thus, the probability that the largest segment is at least three times the shorter is

$$\Pr(E_1)\Pr(1 - X > 3X \mid E_1) + \Pr(E_2)\Pr(X > 3(1 - X) \mid E_2) = \frac{1}{2}.$$

□

14. Let X have pdf $f_{_X}(x) = \begin{cases} x^2/9, & \text{if } 0 < x < 3; \\ 0, & \text{elsewhere.} \end{cases}$

    Find the pdf of $Y = X^3$.

    ***Solution**:* First, compute the cdf of Y

    $$F_{_Y}(y) = \Pr(Y \leqslant y). \tag{19}$$

    Observe that, since $Y = X^3$ and the possible values of X range from 0 to 3, the values of Y will range from 0 to 27. Thus, in the calculations that follow, we will assume that $0 < y < 27$.
    From (19) we get that

    $$\begin{aligned} F_{_Y}(y) &= \Pr(X^3 \leqslant y) \\ &= \Pr(X \leqslant y^{1/3}) \\ &= F_{_X}(y^{1/3}) \end{aligned}$$

    Thus, for $0 < y < 27$, we have that

    $$f_{_Y}(y) = f_{_X}(y^{1/3}) \cdot \frac{1}{3} y^{-3/2}, \tag{20}$$

    where we have applied the Chain Rule. It follows from (20) and the definition of $f_{_X}$ that

    $$f_{_Y}(y) = \frac{1}{9}\left[y^{1/3}\right]^2 \cdot \frac{1}{3} y^{-3/2} = \frac{1}{27}, \quad \text{for } 0 < y < 27. \tag{21}$$

    Combining (21) and the definition of $f_{_X}$ we obtain the pdf for Y:

    $$f_{_Y}(y) = \begin{cases} \dfrac{1}{27}, & \text{for } 0 < y < 27; \\ \\ 0 & \text{elsewhere;} \end{cases}$$

    in other words $Y \sim \text{Uniform}(0, 27)$. □