Solutions to Review Problems for Exam 2

1. Let X and Y be independent Normal(0,1) random variables. Put $Z = \dfrac{Y}{X}$. Compute the distribution functions $F_Z(z)$ and $f_Z(z)$.

   Solution: Since $X, Y \sim$ Normal(0,1), their pdfs are given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, \quad \text{for } x \in \mathbb{R},$$

and

$$f_Y(y) = \frac{1}{\sqrt{2\pi}}\, e^{-y^2/2}, \quad \text{for } y \in \mathbb{R},$$

respectively. The joint pdf of (X, Y) is then

$$f_{(X,Y)}(x,y) = \frac{1}{2\pi}\, e^{-(x^2+y^2)/2}, \quad \text{for } (x,y) \in \mathbb{R}^2. \tag{1}$$

We compute the cdf of Z,

$$F_Z(z) = \Pr(Z \leqslant z) = \Pr\left(\frac{y}{x} \leqslant z\right),$$

or

$$F_Z(z) = \iint_{\frac{y}{x}\leqslant z} f_{(X,Y)}(x,y)\; dxdy, \tag{2}$$

where the integrand in (2) is given in (1) and the integration is done over the region

$$R = \left\{(x,y) \in \mathbb{R}^2 \mid \frac{y}{x} \leqslant z\right\}.$$

Make the change variables

$$\begin{aligned} u &= x \\ v &= \frac{y}{x}, \end{aligned}$$

so that

$$\begin{aligned} x &= u \\ y &= uv, \end{aligned} \tag{3}$$

in the integral in (2) to obtain

$$F_Z(z) = \int_{-\infty}^{z}\int_{-\infty}^{\infty} f_{(X,Y)}(u, uv)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\; dudv, \tag{4}$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 1 & 0 \\ v & u \end{pmatrix} = u, \tag{5}$$

is the Jacobian determinant of the transformation in (3). It then follows from (4) and (5) that

$$F_Z(z) = \int_{-\infty}^{z}\int_{-\infty}^{\infty} f_{(X,Y)}(u, uv)\,|u|\;dudv, \tag{6}$$

Differentiating with respect to z and using the definition of the joint pdf of (X, Y) in (1) we obtain from (6) that

$$f_Z(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |u|\; e^{-(1+z^2)u^2/2}\;du, \tag{7}$$

where we have also used the Fundamental Theorem of Calculus.

Since the integrand in (7) is an even function of u, we can rewrite the expression for f_Z in (7) as

$$f_Z(z) = \frac{1}{\pi}\int_{0}^{\infty} u\; e^{-(1+z^2)u^2/2}\;du. \tag{8}$$

Integrating the right–hand side of equation in (8) we obtain

$$f_Z(z) = \frac{1}{\pi}\cdot\frac{1}{1+z^2}, \qquad \text{for } z \in \mathbb{R}. \tag{9}$$

The cdf of Z is then obtained by integrating (9) to get

$$F_Z(z) = \int_{-\infty}^{z} f_Z(z)\;dz = \frac{1}{2} + \frac{1}{\pi}\arctan(z), \qquad \text{for } z \in \mathbb{R}.$$

□

2. A random point (X, Y) is distributed uniformly on the square with vertices $(-1, -1)$, $(1, -1)$, $(1, 1)$ and $(-1, 1)$.

   (a) Give the joint pdf for X and Y.

   (b) Compute the following probabilities:

      (i) $\Pr(X^2 + Y^2 < 1)$,

      (ii) $\Pr(2X - Y > 0)$,

      (iii) $\Pr(|X + Y| < 2)$.

***Solution**:* The square is pictured in Figure 1 and has area 4.

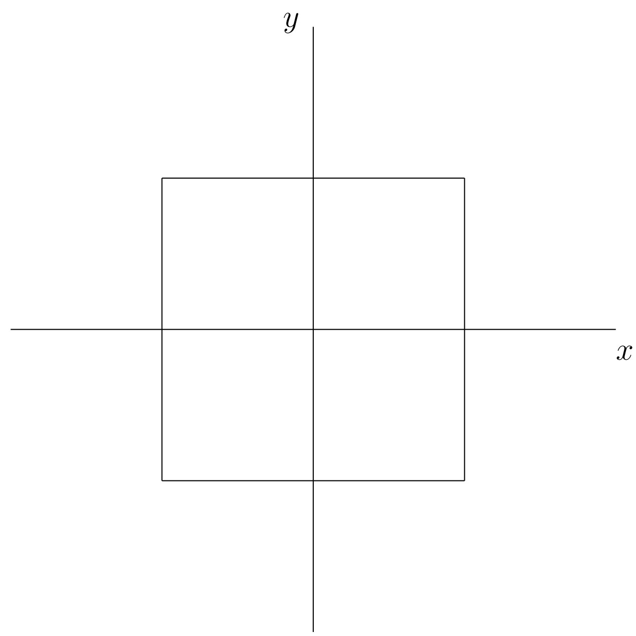

Figure 1: Sketch of square in Problem 2

(a) Consequently, the joint pdf of (X, Y) is given by

$$f_{(X,Y)}(x,y) = \begin{cases} \dfrac{1}{4}, & \text{for } -1 < x < 1, -1 < y < 1; \\ \\ 0 & \text{elsewhere.} \end{cases} \tag{10}$$

(b) Denoting the square in Figure 1 by R, it follows from (10) that, for any subset A of $\mathbb{R}^2$,

$$\Pr[(x,y) \in A] = \iint_A f_{(X,Y)}(x,y)\; dxdy = \frac{1}{4} \cdot \text{area}(A \cap R); \tag{11}$$

that is, $\Pr[(x,y) \in A]$ is one–fourth the area of the portion of A in R.
We will use the formula in (11) to compute each of the probabilities in (i), (ii) and (iii).

(i) In this case, A is the circle of radius 1 around the origin in $\mathbb{R}^2$ and pictured in Figure 2.
Note that the circle A in Figure 2 is entirely contained in the square R so that, by the formula in (11),

$$\Pr(X^2 + Y^2 < 1) = \frac{\text{area}(A)}{4} = \frac{\pi}{4}.$$

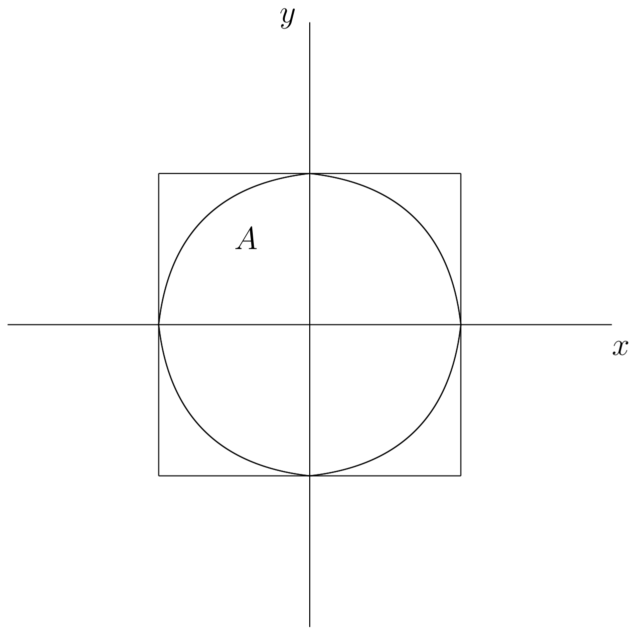

Figure 2: Sketch of A in Problem 2(i)

(ii) The set A in this case is pictured in Figure 3 on page 5. Thus, in this case, $A \cap R$ is a trapezoid of area $2 \cdot \dfrac{\frac{1}{2} + \frac{3}{2}}{2} = 2$, so that, by the formula in (11),

$$\Pr(2X - Y > 0) = \frac{1}{4} \cdot \text{area}(A \cap R) = \frac{1}{2}.$$

(iii) In this case, A is the region in the xy–plane between the lines $x+y = 2$ and $x + y = -2$ (see Figure 4 on page 6). Thus, $A \cap R$ is R so that, by the formula in (11),

$$\Pr(|X + Y| < 2) = \frac{\text{area}(R)}{4} = 1.$$

□

3. Prove that if the joint cdf of X and Y satisfies

$$F_{(X,Y)}(x, y) = F_X(x)F_Y(y),$$

then for any pair of intervals (a, b) and (c, d),

$$\Pr(a < X \leq b, c < Y \leq d) = \Pr(a < X \leqslant b)\Pr(c < Y \leqslant d).$$

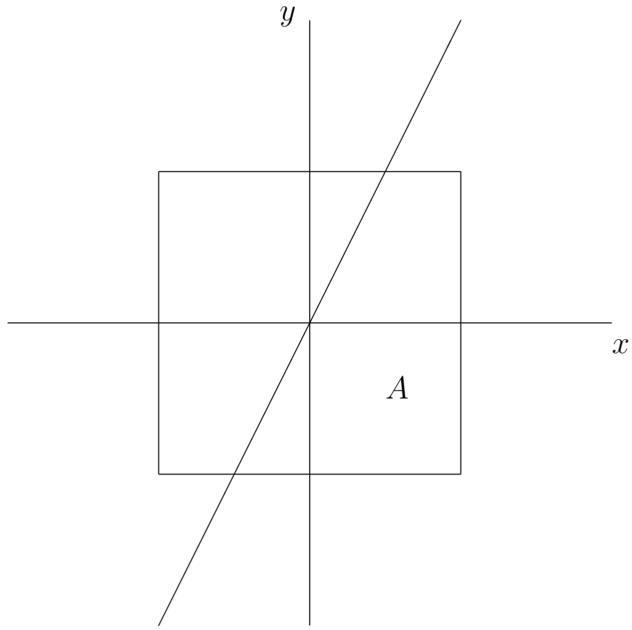

Figure 3: Sketch of A in Problem 2(ii)

***Solution**:* First show that

$$\Pr(a < X \leq b, c < Y \leq d) = F_{(X,Y)}(b,d) - F_{(X,Y)}(b,c) - F_{(X,Y)}(a,d) + F_{(X,Y)}(a,c)$$

(see Problem 1 in Assignment #15). Then,

$$\begin{aligned}
\Pr(a < X \leq b, c < Y \leq d) &= F_X(b)F_Y(d) - F_X(b)F_Y(c) \\
&\quad - F_X(a)F_Y(d) + F_X(a)F_Y(c) \\
&= (F_X(b) - F_X(a))F_Y(d) \\
&\quad - (F_X(b) - F_X(a))F_Y(c) \\
&= (F_X(b) - F_X(a))(F_Y(d) - F_Y(c)) \\
&= \Pr(a < X \leqslant b)\Pr(c < Y \leqslant d),
\end{aligned}$$

which was to be shown. □

4. The random pair (X, Y) has the joint distribution shown in Table 1 on page 6.

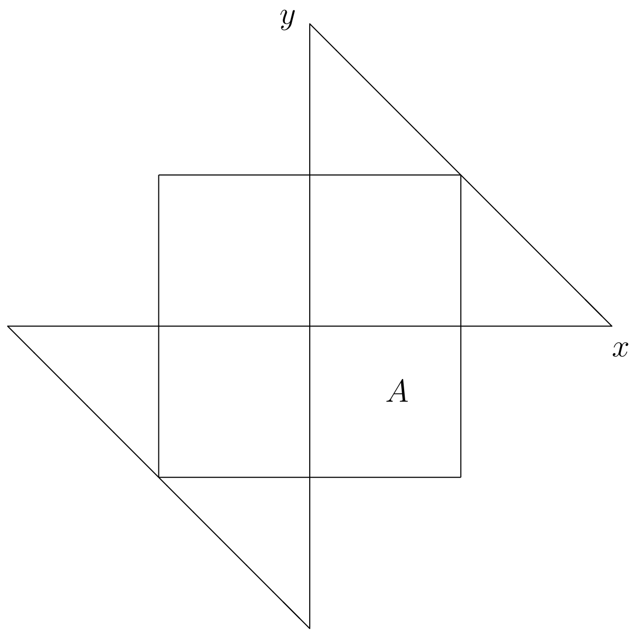

Figure 4: Sketch of A in Problem 2(iii)

$X\backslash Y$	2	3	4
1	$\frac{1}{12}$	$\frac{1}{6}$	0
2	$\frac{1}{6}$	0	$\frac{1}{3}$
3	$\frac{1}{12}$	$\frac{1}{6}$	0

Table 1: Joint Probability Distribution for X and Y, $p_{(X,Y)}$

(a) Show that X and Y are not independent.

***Solution**:* Table 2 shows the marginal distributions of X and Y on the margins on page 7.

Observe from Table 2 that

$$p_{(X,Y)}(1,4) = 0,$$

while

$$p_X(1) = \frac{1}{4} \quad \text{and} \quad p_Y(4) = \frac{1}{3}.$$

Thus,

$$p_X(1) \cdot p_Y(4) = \frac{1}{12};$$

$X\backslash Y$	2	3	4	p_X
1	$\frac{1}{12}$	$\frac{1}{6}$	0	$\frac{1}{4}$
2	$\frac{1}{6}$	0	$\frac{1}{3}$	$\frac{1}{2}$
3	$\frac{1}{12}$	$\frac{1}{6}$	0	$\frac{1}{4}$
p_Y	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

Table 2: Joint pdf for X and Y and marginal distributions p_X and p_Y

so that

$$p_{(X,Y)}(1,4) \neq p_X(1) \cdot p_Y(4),$$

and, therefore, X and Y are not independent. □

(b) Give a probability table for random variables U and V that have the same marginal distributions as X and Y, respectively, but are independent.

***Solution**:* Table 3 on page 7 shows the joint pmf of (U, V) and the marginal distributions, p_U and p_V. □

$U\backslash V$	2	3	4	p_U
1	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
2	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$
3	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
p_V	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

Table 3: Joint pdf for U and V and their marginal distributions.

5. Let X denote the number of trials needed to obtain the first head, and let Y be the number of trials needed to get two heads in repeated tosses of a fair coin. Are X and Y independent random variables?

***Solution**:* X has a geometric distribution with parameter $p = \frac{1}{2}$, so that

$$p_X(k) = \frac{1}{2^k}, \quad \text{for } k = 1, 2, 3, \ldots \tag{12}$$

On the other hand,

$$\Pr[Y = 2] = \frac{1}{4}, \tag{13}$$

since, in two repeated tosses of a coin, the events are HH, HT, TH and TT, and these events are equally likely.

Next, consider the joint event $(X = 2, Y = 2)$. Note that

$$(X = 2, Y = 2) = [X = 2] \cap [Y = 2] = \emptyset,$$

since $[X = 2]$ corresponds to the event TH, while $[Y = 2]$ to the event HH. Thus,

$$\Pr(X = 2, Y = 2) = 0,$$

while

$$p_X(2) \cdot p_Y(2) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16},$$

by (12) and (13). Thus,

$$p_{(X,Y)}(2, 2) \neq p_X(2) \cdot p_X(2).$$

Hence, X and Y are not independent. □

6. Let $X \sim \text{Normal}(0, 1)$ and put $Y = X^2$. Find the pdf for Y.

***Solution**:* The pdf of X is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \text{for } -\infty < x < \infty.$$

We compute the pdf for Y by first determining its cdf:

$$\begin{aligned} \Pr(Y \leqslant y) &= P(X^2 \leqslant y) \quad \text{for } y \geqslant 0 \\ &= \Pr(-\sqrt{y} \leqslant X \leqslant \sqrt{y}) \\ &= \Pr(-\sqrt{y} < X \leqslant \sqrt{y}), \quad \text{since X is continuous.} \end{aligned}$$

Thus,

$$\begin{aligned} \Pr(Y \leqslant y) &= \Pr(X \leqslant \sqrt{y}) - \Pr(X \leqslant -\sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \quad \text{for } y > 0. \end{aligned}$$

We then have that the cdf of Y is

$$F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) \quad \text{for } y > 0,$$

from which we get, after differentiation with respect to y,

$$\begin{aligned} f_Y(y) &= F'_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} + F'_X(-\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} \\ &= f_X(\sqrt{y}) \frac{1}{2\sqrt{y}} + f_X(-\sqrt{y}) \frac{1}{2\sqrt{y}} \\ &= \frac{1}{2\sqrt{y}} \left\{ \frac{1}{\sqrt{2\pi}}\, e^{-y/2} + \frac{1}{\sqrt{2\pi}}\, e^{-y/2} \right\} \\ &= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{y}}\, e^{-y/2}, \end{aligned}$$

for $y > 0$, where we have applied the Chain Rule. Hence,

$$f_Y(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} \cdot \dfrac{1}{\sqrt{y}}\; e^{-y/2}, & \text{for } y > 0; \\ \\ 0 & \text{for } y \leqslant 0. \end{cases}$$

□

7. Let X and Y be independent Normal$(0,1)$ random variables. Compute

$$P(X^2 + Y^2 < 1).$$

***Solution**:* Since $X, Y \sim$ Normal$(0,1)$, their pdfs are given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}}\; e^{-x^2/2}, \quad \text{for } x \in \mathbb{R},$$

and

$$f_Y(y) = \frac{1}{\sqrt{2\pi}}\; e^{-y^2/2}, \quad \text{for } y \in \mathbb{R},$$

respectively. The joint pdf of (X,Y) is then

$$f_{(X,Y)}(x,y) = \frac{1}{2\pi}\; e^{-(x^2+y^2)/2}, \quad \text{for } (x,y) \in \mathbb{R}^2. \tag{14}$$

Thus,

$$P(X^2 + Y^2 < 1) = \iint_{x^2+y^2<1} f_{(X,Y)}(x,y)\; dxdy, \tag{15}$$

where the integrand is given in (14) and the integral in (15) is evaluated over the disc of radius 1 centered around the origin in $\mathbb{R}^2$.

We evaluate the integral in (15) by changing to polar coordinates to get

$$\begin{aligned} P(X^2 + Y^2 < 1) &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 e^{-r^2/2} r\,dr\,d\theta \\ &= \int_0^1 e^{-r^2/2} r\,dr \\ &= \left[-e^{-r^2/2}\right]_0^1 \\ &= 1 - e^{-1/2}, \end{aligned}$$

or $\Pr(X^2+Y^2<1) = 1 - \dfrac{1}{\sqrt{e}}$. □

8. Suppose that X and Y are independent random variables such that $X \sim \text{Uniform}(0,1)$ and $Y \sim \text{Exponential}(1)$.

(a) Let $Z = X+Y$. Find F_Z and f_Z.

***Solution**:* Since $X \sim \text{Uniform}(0,1)$ and $Y \sim \text{Exponential}(1)$, their pdfs are given by

$$f_X(x) = \begin{cases} 1 & \text{if } 0 < x < 1; \\ 0 & \text{elsewhere,} \end{cases}$$

and

$$f_Y(y) = \begin{cases} e^{-y} & \text{if } y > 0; \\ 0 & \text{if } y \leqslant 0, \end{cases}$$

respectively. The joint pdf of (X,Y) is then

$$f_{(X,Y)}(x,y) = \begin{cases} e^{-y} & \text{if } 0 < x < 1,\ y > 0; \\ 0 & \text{elsewhere.} \end{cases} \tag{16}$$

We compute the cdf of Z,

$$F_Z(z) = \Pr(X \leqslant u) = \Pr\left(X+Y \leqslant z\right),$$

or

$$F_U(u) = \iint_{x+y\leqslant z} f_{(X,Y)}(x,y)\, dxdy, \tag{17}$$

where the integrand in (17) is given in (16) and the integration is done over the region

$$R = \left\{(x,y) \in \mathbb{R}^2 \mid x+y \leqslant z\right\}.$$

Make the change variables

$$\begin{aligned} u &= x \\ v &= x+y, \end{aligned}$$

so that

$$\begin{aligned} x &= u \\ y &= v-u, \end{aligned} \tag{18}$$

in the integral in (17) to obtain

$$F_Z(z) = \int_{-\infty}^{z} \int_{-\infty}^{\infty} f_{(X,Y)}(u, v-u) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, dudv, \tag{19}$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = 1, \tag{20}$$

is the Jacobian determinant of the transformation in (18). It then follows from (19) and (20) that

$$F_Z(z) = \int_{-\infty}^{z} \int_{-\infty}^{\infty} f_{(X,Y)}(u, v-u) \, dudv. \tag{21}$$

Differentiating with respect to z and using the definition of the joint pdf of (X, Y) in (16) we obtain from (21) that

$$f_Z(z) = \int_{-\infty}^{\infty} f_{(X,Y)}(u, z-u) \, du. \tag{22}$$

where we have also used the Fundamental Theorem of Calculus.

Next, use the definition of $f_{(X,Y)}$ in (16) to rewrite (22) as

$$f_Z(z) = \int_{0}^{1} f_{(X,Y)}(u, z-u) \, du, \quad \text{for } z > 0, \tag{23}$$

We consider two cases, (i) $0 < z \leqslant 1$, and (ii) $z > 1$.

(i) For $0 < z \leqslant 1$, use (16) to obtain from (23) that

$$\begin{aligned} f_Z(z) &= \int_0^z e^{u-z} \, du \\ &= e^{-z} \int_0^z e^{u} \, du \\ &= 1 - e^{-z}, \end{aligned}$$

so that

$$f_Z(z) = 1 - e^{-z}, \quad \text{for } 0 < z \leqslant 1. \tag{24}$$

(ii) For $z > 0$, use (16) to obtain from (23) that

$$\begin{aligned} f_Z(z) &= \int_0^1 e^{u-z}\, du \\ &= e^{-z}\int_0^1 e^{u}\, du \\ &= (e-1)e^{-z}, \end{aligned}$$

so that

$$f_Z(z) = (e-1)e^{-z}, \qquad \text{for } z > 1. \tag{25}$$

Combining (24) and (25) we obtain the cdf

$$f_Z(z) = \begin{cases} 0 & \text{for } z \leqslant 0; \\ 1 - e^{-z}, & \text{for } 0 < z \leqslant 1; \\ (e-1)e^{-z}, & \text{for } z > 1. \end{cases} \tag{26}$$

Finally, integrating (26) yields the cdf

$$F_Z(z) = \begin{cases} 0 & \text{for } z \leqslant 0; \\ z + e^{-z} - 1, & \text{for } 0 < z \leqslant 1; \\ e^{-1} + (e-1)(e^{-1} - e^{-z}), & \text{for } z > 1. \end{cases}$$

□

(b) Let $U = Y/X$. Find F_U and f_U.

***Solution**:* Since $X \sim \text{Uniform}(0,1)$ and $Y \sim \text{Exponential}(1)$, their pdfs are given by

$$f_X(x) = \begin{cases} 1 & \text{if } 0 < x < 1; \\ 0 & \text{elsewhere}, \end{cases}$$

and

$$f_Y(y) = \begin{cases} e^{-y} & \text{if } y > 0; \\ 0 & \text{if } y \leqslant 0, \end{cases}$$

respectively. The joint pdf of (X, Y) is then

$$f_{(X,Y)}(x,y) = \begin{cases} e^{-y} & \text{if } 0 < x < 1,\ y > 0; \\ 0 & \text{elsewhere}. \end{cases} \tag{27}$$

We compute the cdf of U,

$$F_U(u) = \Pr(U \leqslant u) = \Pr\left(\frac{Y}{X} \leqslant u\right),$$

or

$$F_U(u) = \iint_{\frac{y}{x} \leqslant u} f_{(X,Y)}(x,y)\; dxdy, \tag{28}$$

where the integrand in (28) is given in (27) and the integration is done over the region

$$R = \left\{(x,y) \in \mathbb{R}^2 \mid \frac{y}{x} \leqslant u\right\}.$$

Make the change variables

$$\begin{aligned} w &= x \\ v &= \frac{y}{x}, \end{aligned}$$

so that

$$\begin{aligned} x &= w \\ y &= wv, \end{aligned} \tag{29}$$

in the integral in (28) to obtain

$$F_U(u) = \int_{-\infty}^{u} \int_{-\infty}^{\infty} f_{(X,Y)}(w, wv) \left|\frac{\partial(x,y)}{\partial(w,v)}\right| \; dwdv, \tag{30}$$

where

$$\frac{\partial(x,y)}{\partial(w,v)} = \det\begin{pmatrix} 1 & 0 \\ v & w \end{pmatrix} = w, \tag{31}$$

is the Jacobian determinant of the transformation in (29). It then follows from (30) and (31) that

$$F_U(u) = \int_{-\infty}^{u} \int_{-\infty}^{\infty} f_{(X,Y)}(w, wv)\, |w| \; dwdv. \tag{32}$$

Differentiating with respect to u and using the definition of the joint pdf of (X,Y) in (27) we obtain from (32) that

$$f_U(u) = \int_{-\infty}^{\infty} f_{(X,Y)}(w, wu)|w| \; dw. \tag{33}$$

where we have also used the Fundamental Theorem of Calculus.
Next, use the definition of $f_{(X,Y)}$ in (27) to rewrite (33) as

$$f_U(u) = \int_{0}^{1} e^{-uw} w \; dw, \quad \text{for } u > 1, \tag{34}$$

We evaluate the integral in (34) by integration by parts to get

$$\begin{aligned} f_{_U}(u) &= \left[-\frac{w}{u}\, e^{-uw} - \frac{1}{u^2}\, e^{-uw}\right]_0^1 \\ &= \frac{1}{u^2} - \frac{1}{u}\, e^{-u} - \frac{1}{u^2}\, e^{-u}, \qquad \text{for } u > 0. \end{aligned} \tag{35}$$

In order to compute the cdf, $F_{_U}$, we can integrate (28) in Cartesian coordinates to get

$$\begin{aligned} F_{_U}(u) &= \int_0^1 \int_0^{ux} e^{-y}\, dydx \\ &= \int_0^1 [1 - e^{-ux}]\, dx \\ &= 1 + \frac{1}{u}[e^{-u} - 1], \end{aligned}$$

so that

$$F_{_U}(u) = \begin{cases} 1 + \dfrac{1}{u}[e^{-u} - 1], & \text{for } u > 0; \\ \\ 0 & \text{for } u \leqslant 0. \end{cases} \tag{36}$$

Note that differentiating $F_{_U}(u)$ in (36) with respect to u, for $u > 0$, leads to (35). We then have that

$$f_{_U}(u) = \begin{cases} \dfrac{1}{u^2}(1 - e^{-u}) - \dfrac{1}{u}\, e^{-u}, & \text{for } u > 0; \\ \\ 0 & \text{for } u \leqslant 0. \end{cases}$$

□

9. Let $X \sim \text{Exponential}(1)$, and define Y to be the integer part of $X + 1$; that is, $Y = i+1$ if and only if $i \leqslant X < i+1$, for $i = 0, 1, 2, \ldots$ Find the pmf of Y, and deduce that $Y \sim \text{Geometric}(p)$ for some $0 < p < 1$. What is the value of p?

   ***Solution**:* Compute

$$\Pr[Y = i+1] = \Pr[i \leqslant X < i+1] = \Pr[i < X \leqslant i+1],$$

since X is continuous; so that

$$\Pr[Y = i+1] = \int_i^{i+1} f_X(x)\, dx, \tag{37}$$

where

$$f_X(x) = \begin{cases} e^{-x} & \text{if } x > 0; \\ 0 & \text{if } x \leqslant 0, \end{cases} \tag{38}$$

since $X \sim \text{Exponential}(1)$.

Evaluating the integral in (37), for $i \geqslant 0$ and f_X as given in (38), yields

$$\begin{aligned} \Pr[Y = i+1] &= \int_i^{i+1} e^{-x}\, dx \\ &= \left[-e^{-x}\right]_i^{i+1} \\ &= e^{-i} - e^{-i-1}, \end{aligned}$$

so that

$$\Pr[Y = i+1] = \left(\frac{1}{e}\right)^i \left(1 - \frac{1}{e}\right) \tag{39}$$

It follows from (39) that $Y \sim \text{Geometric}(p)$ with $p = 1 - \dfrac{1}{e}$. □

10. The expected number of typographical errors on a page of a certain magazine is 0.20. What is the probability that an article of 10 pages contains (a) no typographical errors, and (b) two or more typographical errors. Explain your reasoning.

    ***Solution**:* Let X denote the number of typographical errors in one page. Then, X may be modeled by a Poisson random variable with parameter λ, where $E(X) = \lambda$, and $\lambda = 0.20$ in this problem. We then have that the pmf of X is

$$p_X(k) = \frac{\lambda^k}{k!}\, e^{-\lambda}, \quad \text{for } k = 0, 1, 2, 3, \ldots \tag{40}$$

The moment generating function of X is

$$\psi_X(t) = e^{\lambda(e^t - 1)}, \quad \text{for } t \in \mathbb{R}. \tag{41}$$

Let $X_1, X_2, \ldots, X_n$, where $n = 10$, denote the number of typographical errors in pages 1 through n, respectively. We may assume that $X_1, X_2, \ldots X_n$ are independent and that they all have a Poisson(λ) distribution.

Put $Y = X_1 + X_2 + \cdots + X_n$. Then Y gives the number of typographical errors in the n pages of the article. The moment generating function of Y is then

$$\psi_{_Y}(t) = \psi_{_{X_1}}(t) \cdot \psi_{_{X_2}}(t) \cdots \psi_{_{X_n}}(t), \quad \text{for } t \in \mathbb{R}, \tag{42}$$

by the independence of the X_i's. It then follows from (42) and (41) that

$$\psi_{_Y}(t) = [e^{\lambda(e^t-1)}]^n \quad \text{for } t \in \mathbb{R},$$

or

$$\psi_{_Y}(t) = e^{n\lambda(e^t-1)} \quad \text{for } t \in \mathbb{R}. \tag{43}$$

Comparing (43) with (41), we see that Y has a Poisson distribution with parameter $n\lambda$; that is,

$$Y \sim \text{Poisson}(n\lambda),$$

so that, in view of (40),

$$p_{_Y}(k) = \frac{(n\lambda)^k}{k!}\, e^{-n\lambda}, \quad \text{for } k = 0, 1, 2, 3, \ldots \tag{44}$$

In this problem $n = 10$ and $\lambda = 0.2$.

(a) The probability that the article contains no typographical errors is

$$\Pr[Y = 0] = e^{-n\lambda} = e^{-2} \approx 13.53\%.$$

(b) The probability that the article contains two or more typographical errors is

$$\begin{aligned} \Pr[Y \geqslant 2] &= 1 - \Pr[Y \leqslant 1] \\ &= 1 - \Pr[Y = 0] - \Pr[Y = 1] \\ &= 1 - e^{-n\lambda} - n\lambda e^{-n\lambda} \\ &= 1 - e^{-2} - 2e^{-2} \\ &\approx 59.4\%. \end{aligned}$$

□