Solutions to Review Problems for Final Exam

1. Three cards are in a bag. One card is red on both sides. Another card is white on both sides. The third card in red on one side and white on the other side. A card is picked at random and placed on a table. Compute the probability that if a given color is shown on top, the color on the other side is the same as that of the top.

   Solution: Each card has a likelihood of 1/3 of bing picked.

   Assume for definiteness that the top of the picked card is red. Let T_r denote the event that the top of picked car shows red and B_r denote the event that the bottom of the card is also red. We want to compute

   $$\Pr(B_r \mid T_r) = \frac{\Pr(T_r \cap B_r)}{\Pr(T_r)}. \tag{1}$$

   Note that

   $$\Pr(T_r \cap B_r) = \frac{1}{3}, \tag{2}$$

   since there is only one card for which both sides are red.

   In order to compute $\Pr(T_r)$ observe that there are three equally likely choices out of six for the top of the card to show red; thus,

   $$\Pr(T_r) = \frac{1}{2}. \tag{3}$$

   Hence, using (2) and (3), we obtain from (1) that

   $$\Pr(B_r \mid T_r) = \frac{2}{3}. \tag{4}$$

   Similar calculations can be used to show that

   $$\Pr(T_w) = \frac{1}{2}, \tag{5}$$

   and

   $$\Pr(B_w \mid T_w) = \frac{2}{3}. \tag{6}$$

   Let E denote the event that a card showing a given color on the top side will have the same color on the bottom side. Then, by the law of total probability,

   $$\Pr(E) = \Pr(T_r) \cdot \Pr(B_r \mid T_r) + \Pr(T_w) \cdot \Pr(B_w \mid T_w), \tag{7}$$

so that, using (2), (4), (5) and (6), we obtain from (7) that

$$\Pr(E) = \frac{1}{2}\cdot\frac{2}{3} + \frac{1}{2}\cdot\frac{2}{3} = \frac{2}{3}.$$

□

2. Suppose that $0 < \rho < 1$ and let $p(n) = \rho^n(1-\rho)$ for $n = 0, 1, 2, 3, \ldots$

   (a) Verify that p is the probability mass function (pmf) for a random variable.
   Solution: Compute

   $$\sum_{n=0}^{\infty} \rho^n(1-\rho) = (1-\rho)\sum_{n=0}^{\infty}\rho^n = (1-\rho)\cdot\frac{1}{1-\rho} = 1,$$

   since $0 < \rho < 1$ and, therefore, the geometric series $\sum_{n=0}^{\infty}\rho^n$ converges to $\dfrac{1}{1-\rho}$. □

   (b) Let X denote a discrete random variable with pmf p. Compute $P_X(X > 1)$.
   Solution: Compute

   $$\begin{aligned} P_X(X > 1) &= 1 - P_X(X \leqslant 1) \\ &= 1 - p(0) - p(1) \\ &= 1 - (1-\rho) - \rho(1-\rho) \\ &= \rho^2. \end{aligned}$$

   □

3. If the pdf of a random variable X is

$$f(x) = \begin{cases} 2xe^{-x^2}, & x > 0; \\ 0, & x \leq 0 \end{cases} \qquad (8)$$

Find the pdf of $Y = X^2$.

Solution: First, compute the cdf of Y:

$$\begin{aligned} F_{_Y}(y) &= \Pr(Y \leqslant y), \qquad \text{for } y > 0, \\ &= \Pr(X^2 \leqslant y) \\ &= \Pr(|X| \leqslant \sqrt{y}) \\ &= \Pr(-\sqrt{y} \leqslant X \leqslant \sqrt{y}), \end{aligned}$$

so that

$$F_{_Y}(y) = \Pr(-\sqrt{y} < X \leqslant \sqrt{y}), \qquad \text{for } y > 0. \tag{9}$$

since X is a continuous random variable.

It follows from (9) and the definition of the cdf of X that

$$F_{_Y}(y) = F_{_X}(\sqrt{y}) - F_{_X}(-\sqrt{y}), \qquad \text{for } y > 0. \tag{10}$$

Differentiating $F_{_Y}$ in (10) with respect to y yields

$$f_{_Y}(y) = f(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} + f(-\sqrt{y}) \cdot \frac{1}{2\sqrt{y}}, \qquad \text{for } y > 0, \tag{11}$$

where we have applied the Chain Rule and f is given in (8). It follows from (11) and the definition of f in (8) that

$$f_{_Y}(y) = \begin{cases} e^{-y}, & \text{for } y > 0; \\ 0, & \text{for } y \leqslant 0, \end{cases}$$

so that Y has an Exponential(1) distribution. □

4. Let $N(t)$ denote the number of mutations in a bacterial colony that occur during the interval $[0, t)$. Assume that $N(t) \sim \text{Poisson}(\lambda t)$ where $\lambda > 0$ is a positive parameter.

   (a) Give an interpretation for λ.

   Answer: λ is the average number of mutations per unit of time. □

   (b) Let T_1 denote the time that the first mutation occurs. Find the distribution of T_1.

   Solution: Observe that, for $t > 0$, the event $[T_1 > t]$ is the same as the event $[N(t) = 0]$; that is, if $t < T_1$, there have have been mutations in the time interval $[0, t]$. Consequently,

$$\Pr[T_1 > t] = \Pr[N(t) = 0] = e^{-\lambda t},$$

since $N(t) \sim \text{Poisson}(\lambda t)$. Thus,

$$\Pr[T_1 \leqslant t] = 1 - \Pr[T_1 > t] = 1 - e^{-\lambda t}, \quad \text{for } t > 0.$$

We then have that the cdf of T_1 is

$$F_{T_1}(t) = \begin{cases} 1 - e^{-\lambda t}, & \text{for } t > 0; \\ 0 & \text{for } t \leqslant 0. \end{cases} \tag{12}$$

It follows from (12) that the pdf for T_1 is

$$f_{T_1}(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{for } t > 0; \\ 0 & \text{for } t \leqslant 0, \end{cases}$$

which is the pdf for an exponential distribution with parameter $\beta = 1/\lambda$; thus,

$$T_1 \sim \text{Exponential}(1/\lambda).$$

□

5. Two checkers at a service station complete checkouts independent of one another in times $T_1 \sim \text{Exponential}(\mu_1)$ and $T_2 \sim \text{Exponential}(\mu_2)$, respectively. That is, one checker serves $1/\mu_1$ customers per unit time on average, while the other serves $1/\mu_2$ customers per unit time on average.

   (a) Give the joint pdf, $f_{T_1,T_2}(t_1, t_2)$, of T_1 and T_2.
   ***Solution**:* Since T_1 and T_2 are independent random variables, the joint pdf of (T_1, T_2) is given by

   $$f_{(T_1,T_2)}(x, y) = f_{T_1}(x) \cdot f_{T_1}(y), \quad \text{for } (x, y) \in \mathbb{R}^2, \tag{13}$$

   where

   $$f_{T_1}(x) = \begin{cases} \dfrac{1}{\mu_1}\, e^{-x/\mu_1}, & \text{for } x > 0; \\ \\ 0 & \text{for } x \leqslant 0, \end{cases} \tag{14}$$

   and

   $$f_{T_2}(y) = \begin{cases} \dfrac{1}{\mu_2}\, e^{-y/\mu_2}, & \text{for } y > 0; \\ \\ 0 & \text{for } y \leqslant 0. \end{cases} \tag{15}$$

It follows from (13), (13) and (15) that the joint pdf of (T_1, T_2) is

$$f_{(T_1,T_2)}(t_1, t_2) = \begin{cases} \dfrac{1}{\mu_1\mu_2}\, e^{-t_1/\mu_1 - t_2/\mu_2}, & \text{for } t_1 > 0 \text{ and } t_2 > 0; \\ \\ 0 & \text{elsewhere.} \end{cases}$$

□

(b) Define the minimum service time, T_m, to be $T_m = \min\{T_1, T_2\}$. Determine the type of distribution that T_m has and give its pdf, $f_{T_m}(t)$.

Solution: Observe that, for $t > 0$, the event $[T_m > t]$ is the same as the event $[T_1 > t, T_2 > t]$, since T_m is the smallest of T_1 and T_2. Consequently,

$$\Pr[T_m > t] = \Pr[T_1 > t, T_2 > t]. \tag{16}$$

Thus, by the independence of T_1 and T_2, it follows from (16) that

$$\Pr[T_m > t] = \Pr[T_1 > t] \cdot \Pr[T_2 > t], \tag{17}$$

where

$$\Pr[T_i > t] = 1 - F_{T_i}(t) = 1 - (1 - e^{-t/\mu_i}),$$

so that

$$\Pr[T_1 > t] = e^{-t/\mu_i}, \qquad \text{for } t > 0 \text{ and } i = 1, 2. \tag{18}$$

Combining (17) and (18) then yields

$$\Pr[T_m > t] = e^{-t/mu_1 - t/\mu_2},$$

or

$$\Pr[T_m > t] = e^{-t/\beta}, \tag{19}$$

where we have set

$$\frac{1}{\beta} = \frac{1}{\mu_1} + \frac{1}{\mu_2}. \tag{20}$$

It follows from (19) that the cdf of T_m is

$$F_{T_m}(t) = \begin{cases} 1 - e^{-t/\beta}, & \text{for } t > 0; \\ 0 & \text{for } t \leqslant 0, \end{cases}$$

where β is given by (20). Thus, the pdf for T_m is

$$f_{T_m}(t) = \begin{cases} \dfrac{1}{\beta}\, e^{-t/\beta}, & \text{for } t > 0; \\ \\ 0 & \text{for } t \leqslant 0, \end{cases} \tag{21}$$

which is the pdf for an exponential distribution with parameter β given by (20); thus,

$$T_m \sim \text{Exponential}(\beta), \qquad \text{where } \beta = \frac{\mu_1\mu_2}{\mu_1+\mu_2}. \tag{22}$$

□

(c) Suppose that, on average, one of the checkers serves 4 customers in an hour, and the other serves 6 customers per hour. On average, what is the minimum amount of time that a customer will spend being served at the service station?

***Solution**:* We compute the expected value of T_m, where T_m has pdf given in (21) with

$$\beta = \frac{\frac{1}{4}\cdot\frac{1}{6}}{\frac{1}{4}+\frac{1}{6}} = \frac{1}{10},$$

in view of (22). Thus, on average, the minimum time spent by a customer being served at the service station is one tenth of an hour, or 6 minutes.
□