Review Problems for Final Exam

1. Three cards are in a bag. One card is red on both sides. Another card is white on both sides. The third card in red on one side and white on the other side. A card is picked at random and placed on a table. Compute the probability that if a given color is shown on top, the color on the other side is the same as that of the top.

2. Suppose that $0 < \rho < 1$ and let $p(n) = \rho^n(1-\rho)$ for $n = 0, 1, 2, 3, \ldots$

   (a) Verify that p is the probability mass function (pmf) for a random variable.

   (b) Let X denote a discrete random variable with pmf p. Compute $P_X(X > 1)$.

3. If the pdf of a random variable X is

$$f(x) = \begin{cases} 2xe^{-x^2}, & x > 0; \\ 0, & x \leq 0 \end{cases}$$

   Find the pdf of $Y = X^2$.

4. Let $N(t)$ denote the number of mutations in a bacterial colony that occur during the interval $[0, t)$. Assume that $N(t) \sim \text{Poisson}(\lambda t)$ where $\lambda > 0$ is a positive parameter.

   (a) Give an interpretation for λ.

   (b) Let T_1 denote the time that the first mutation occurs. Find the distribution of T_1.

5. Two checkers at a service station complete checkouts independent of one another in times $T_1 \sim \text{Exponential}(\mu_1)$ and $T_2 \sim \text{Exponential}(\mu_2)$, respectively. That is, one checker serves $1/\mu_1$ customers per unit time on average, while the other serves $1/\mu_2$ customers per unit time on average.

   (a) Give the joint pdf, $f_{T_1,T_2}(t_1, t_2)$, of T_1 and T_2.

   (b) Define the minimum service time, T_m, to be $T_m = \min\{T_1, T_2\}$. Determine the type of distribution that T_m has and give its pdf, $f_{T_m}(t)$.

   (c) Suppose that, on average, one of the checkers serves 4 customers in an hour, and the other serves 6 customers per hour. On average, what is the minimum amount of time that a customer will spend being served at the service station?