Assignment #5

Due on Friday, February 10, 2012

Read Section 3.1 on *Modeling Traffic Flow* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`.

Background

In this problem set we consider heat flow in a metal rod. Assume the length of the rod is L and lies along the x–axis with one end at $x = 0$ and the other end at $x = L$. Postulate a temperature function $u(x,t)$ which measures the temperature in the cross section of the rod at x and at time t. Let $\rho(x,t)$ denote the density of the rod in units of mass per volume of the material composing the rod. We also postulate a specific heat function, $c(x,t)$, which measures the heat energy that needs to be supplied to a unit of mass of material to raise its temperature by one unit of temperature. Assume that c is constant and that the cross sectional are of the rod, A, is also constant.

Do the following problems.

1. Give a formula for computing the heat energy, $Q(t)$, contained in the rod in the section between $x = a$ and $x = b$.

2. State a conservation principle that applies to the amount of heat energy in the $[a,b]$ section, assuming that there are no sources of heat in that section.

3. Postulate a heat flux function, $F(x,t)$, which measures the amount of heat energy that flows across a unit of cross sectional area per unit time in the positive x–direction. Re–state the conservation principle in Problem 2 in terms of the heat flux function.

   Assume that the sides of the cylindrical rod are insulated, so that heat can only enter or leave the $[a,b]$ section of the rod through the end sections at a and b.

4. Use the following empirical constitutive equation that relates heat flux to temperature gradient along the rod,

$$F(x,t) = -\kappa \frac{\partial u}{\partial x}(x,t),$$

   where κ is a positive proportionality constant known as the heat conductivity of the material, to re–state the conservation principle obtained in Problem 3.

5. Assuming that ρ and c are constant and that u has continuous partial derivatives up to order 2, derive the partial differential equation

$$c\rho\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\kappa\frac{\partial u}{\partial x}\right) = 0.$$