Review Problems for Exam 1

1. **Modeling the Spread of a Disease.** In a simple model for a disease that is spread through infections transmitted between individuals in a population, the population is divided into three compartments pictured in Figure 1. The

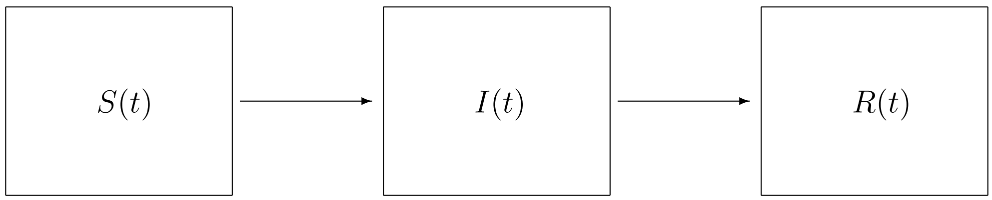

Figure 1: SIR Compartments

first compartment, $S(t)$, denotes the set of individuals in a population that are susceptible to acquiring the disease; the second compartment, $I(t)$, denotes the set of infected individual who can also infect others; and the third compartment, $R(t)$, denotes the set of individuals who had the disease and who have recovered from it; they can no longer get infected.

Assume that the total number of individuals in the population,

$$N = S(t) + I(t) + R(t),$$

is constant. Susceptible individuals can get infected by contact with infectious individuals and move to the infected class. This is indicated by the arrow going from the $S(t)$ compartment to the $I(t)$ compartment.

The rate at which susceptible individuals get infected is proportional to product of number of susceptible individuals and the number of infected individuals with constant of proportionality $\beta > 0$. The rate at which infected individuals recover is proportional to the number of infected individuals with constant of proportionality $\gamma > 0$. What are the units for β and γ?

Use conservation principles to derive a system of differential equations for the functions S, I and R, assuming that they are differentiable. Models of this type were first studied by Kermack and McKendrick in the early 1930s.

Introduce dimensionless variables

$$\widehat{s}(t) = \frac{S(t)}{N}, \quad \widehat{i}(t) = \frac{I(t)}{N}, \quad \widehat{r}(t) = \frac{R(t)}{N}, \quad \text{and} \quad \widehat{t} = \frac{t}{\tau},$$

for some scaling factor, τ, in units of time, in order to write the system in dimensionless form.

2. **Modeling Traffic Flow.** Consider the initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + g'(u)\dfrac{\partial u}{\partial x} &= 0; \\ \\ u(x,0) &= f(x), \end{cases}$$

where $g(u) = u(1-u)$, and the initial condition f is given by

$$f(x) = \begin{cases} 1, & \text{if } x < -1; \\ \dfrac{1}{2}(1-x), & \text{if } -1 \leqslant x < 1; \\ 0, & \text{if } x \geqslant 1. \end{cases}$$

   (a) Sketch the characteristic curves of the partial differential equation.

   (b) Explain how the initial value problem can be solved in this case, and give a formula for $u(x,t)$.

3. **Age Structured Population Models.** Postulate a population density, $n(a,t)$, which also gives the age distribution for individuals in the population; so that, the number of individuals in the population between the ages a_1 and a_2 at time t is given by $\displaystyle\int_{a_1}^{a_2} n(a,t)\ da$.

   (a) Explain why $n(a,t)$ is given in units of population divided by units of time.

   (b) Since a is a function of t, assuming that n is C^1, we can use Chain Rule to compute the rate of change of population density at time t, $\dfrac{dn}{dt}$.
   Explain why
   $$\frac{dn}{dt} = \frac{\partial n}{\partial t} + \frac{\partial n}{\partial a}.$$

   (c) Assume that death rate for individuals of age a in the population is proportional to the number of individuals at that age with constant of proportionality $\mu(a)$.
   Use a conservation principle to derive the following partial differential equation
   $$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial a} = -\mu(a)n$$
   Give the characteristic curves for the equation.

   (d) Give solutions to the partial differential equation derived in the previous part assuming that the death rate is zero for all ages. Interpret your result.