

## Assignment #2

Due on Wednesday, February 6, 2013

Read Section 2.2, *Bacterial Growth in a Chemostat*, in the class lecture notes at <http://pages.pomona.edu/~ajr04747/>.

Read Section 2.3.1 on *Nondimensionalization*, in the class lecture notes at <http://pages.pomona.edu/~ajr04747/>.

**Background and Definitions**

In Section 2.2 in the class lecture notes at <http://pages.pomona.edu/~ajr04747/>, we derived the following system of ordinary differential equations for the chemostat system,

$$\begin{cases} \frac{dn}{dt} = \frac{bnc}{a+c} - \frac{F}{V}n; \\ \frac{dc}{dt} = \frac{F}{V}c_o - \frac{F}{V}c - \frac{\alpha bnc}{a+c}. \end{cases} \quad (1)$$

The variables  $n$  and  $c$  are the bacterial population density and nutrient concentration, respectively, in the chemostat; these are assumed to be differentiable functions of time,  $t$ . The parameter  $c_o$ ,  $F$ ,  $V$ ,  $\alpha$ ,  $a$  and  $b$  have the following interpretations:

- $c_o$  is the nutrient concentration in a reservoir that feeds the chemostat chamber at a constant rate  $F$ ;
- $F$  is also the rate at which culture is drawn from the chemostat chamber;
- $V$  is the fixed volume of the culture;
- $\alpha$  is related to the yield,  $Y = 1/\alpha$ , which is the number of new cells produced in the chemostat due to consumption of one unit of nutrient;
- $b$  is the maximum *per-capita* growth rate allowed by the medium, and  $a$  is the nutrient concentration at which the *per-capita* growth rate is  $b/2$ .

**Do** the following problems

1. Introduce new dimensionless variables

$$\hat{n} = \frac{n}{\mu}, \quad \hat{c} = \frac{c}{a}, \quad \text{and} \quad \tau = \frac{t}{\lambda}, \quad (2)$$

where  $\mu$  and  $\lambda$  are scaling parameters having units of cells/volume and time, respectively.

Verify that the second equation in the system in (1) can be written in the form

$$\frac{d\hat{c}}{d\tau} = \alpha_2 - \frac{\hat{n}\hat{c}}{1 + \hat{c}} - \hat{c},$$

where

$$\alpha_2 = \frac{c_o}{a} \quad (3)$$

and

$$\mu = \frac{a}{\alpha b \lambda}. \quad (4)$$

2. Verify that the parameter  $\alpha_2$  in (3) is dimensionless and that the units of  $\mu$  defined in (4) are indeed cells/volume. Justify your answers.
3. In Section 2.2 in the class lecture notes at <http://pages.pomona.edu/~ajr04747/>, the system in (1) was nondimensionalized to yield the system

$$\begin{cases} \frac{d\hat{n}}{d\tau} = \alpha_1 \frac{\hat{n}\hat{c}}{1 + \hat{c}} - \hat{n}; \\ \frac{d\hat{c}}{d\tau} = \alpha_2 - \frac{\hat{n}\hat{c}}{1 + \hat{c}} - \hat{c}. \end{cases} \quad (5)$$

- (a) Compute the equilibrium solutions of the system in (5) in the  $\hat{n}\hat{c}$ -phase space.
  - (b) Give interpretations for each of the equilibrium points obtained in part (a). Give conditions under which the system in (5) yields biologically feasible equilibrium solutions.
4. Put

$$F(\hat{n}, \hat{c}) = \begin{pmatrix} \alpha_1 \frac{\hat{n}\hat{c}}{1 + \hat{c}} - \hat{n} \\ \alpha_2 - \frac{\hat{n}\hat{c}}{1 + \hat{c}} - \hat{c} \end{pmatrix}$$

Compute the Jacobian matrix,  $DF(\hat{n}, \hat{c})$ , of  $F$ .

5. Compute the eigenvalues of  $DF(\hat{n}, \hat{c})$  at the equilibrium points found in Problem 3 and use this information to determine their stability properties. What do you conclude about the chemostat system?